DOCUMENT RESUME

ED 433 225 SE 062 614

AUTHOR	Bjorkqvist, Ole, Ed.
TITLE	Mathematics Teaching from a Constructivist Point of View. Proceedings of Topic Group 6 at the International Congress on Mathematical Education (8th, Seville, Spain, July 14-21, 1996). Faculty of Education Report No. 3.
INSTITUTION	Abo Akademi Univ., Vasa (Finland). Faculty of Education.
ISBN	ISBN-952-12-0151-7
ISSN	ISSN-1236-0856
PUB DATE	1998-00-00
NOTE	158p.
AVAILABLE FROM	Abo Akademi, Faculty of Education, PB 311, 65101 Vasa, Finland; Tel: +358-(0)6-3247 111; Fax: +358(0)1-3247 302.
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	MF01/PC07 Plus Postage.
DESCRIPTORS	*Constructivism (Learning); Elementary Secondary Education; Foreign Countries; *Learning Theories; *Mathematics Instruction; *Teaching Methods

ABSTRACT

The nationalities of the authors reflect the purpose of the Topic Group, to give an overview of international research on mathematics teaching within a constructivist framework as of 1996. The studies are theoretical as well as empirical, and show a variety of research interests. There is, however, evidence for an emphasis on social aspects within the overall constructivist framework. Social constructivism--together with other similar theoretical starting points--appears to have reached a strong position worldwide within the field of research in mathematics education. Papers include: (1) "Some Questions for Constructivists to Answer" (Ole Bjorkqvist--Finland); (2) "Constructing Mathematics, Learning and Teaching" (Barbara Jaworski--UK); (3) "Constructivism and Constructivist Teaching--Can They Co-exist?" (Carolyn A. Maher--USA); (4) "The Struggles of a 'Constructivist' Curricular Innovation" (Paola Valero--Colombia); (5) "Exploring the Socio-Constructivist Aspects of Maths Teaching: Using 'Tools' in Creating a Maths Learning Culture" (Anna Chronaki--UK); (6) "Study of the Constructive Approach in Mathematics Education" (Tadao Nakahara and Masataka Koyama--Japan); (7) "Basic Imagery and Understandings for Mathematical Concepts" (Peter Bender--Germany); and (8) "Context as Construction" (David Clarke and Sue Helme--Australia). (Author/ASK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

Department of Teacher Education

ED 433 225

Mathematics Teaching from a Constructivist Point of View

Ole Björkqvist (red.)

PERMISSION TO REPRODUCE AND DISSEMINATE THIS MATERIAL HAS BEEN GRANTED BY

C. Wenestam

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

1

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

Reports from the Faculty of Education, Åbo Akademi University
No. 3 1998

BEST COPY AVAILABLE

Department of Teacher Education

Mathematics Teaching from a Constructivist Point of View

Ole Björkqvist (red.)

Reports from the Faculty of Education, Åbo Akademi University
No. 3 1998

Reports from the Faculty of Education are, before accepted for publication, subjected to a referee process.

1998

Printed on recycled paper
© Faculty of Education, the authors

Åbo Akademi
Faculty of Education
Department of Teacher Education

Report No. 3 1998

Björkqvist, Ole, (Ed.)

Mathematics Teaching from a Constructivist Point of View
147 p.

Abstract

This volume contains the full proceedings of Topic Group 6 at the Eighth International Congress on Mathematical Education in Sevilla, Spain, July 14-21, 1996. In addition there are two invited papers related to the theme of the Topic Group, one of which was presented elsewhere at the Congress and the other originally intended for presentation in Topic Group 6.

The nationalities of the authors reflect the purpose of the Topic Group, to give an overview of international research on mathematics teaching within a constructivist framework as of 1996. The studies are theoretical as well as empirical, and show a variety of research interests. There is, however, evidence for an emphasis on social aspects within the overall constructivist framework. Social constructivism – together with other similar theoretical starting points – thus appears to have reached a strong position worldwide within the field of research in mathematics education.

Available from
Åbo Akademi, Faculty of Education, PB 311, 65101 Vasa, Finland
Tel: +358-(0)6-3247 111 Fax: +358-(0)1-3247 302

Åbo Akademi
Pedagogiska fakulteten
Institutionen för lärarutbildning

Rapport nr 3 1998

Björkqvist, Ole, (Ed.)

Mathematics Teaching from a Constructivist Point of View
147 p.

Abstrakt

Denna volym innehåller den fullständiga dokumentationen av aktiviteten inom temagrupp 6 vid den åttonde internationella kongressen om matematikundervisning (ICME8) i Sevilla i Spanien, den 14-21 juli 1996. Dessutom innehåller den två inbjudna artiklar som hänför sig till gruppens tema, av vilka en grundar sig på en presentation på annat ställe under kongressen och den andra ursprungligen var avsedd som en presentation inom temagrupp 6.

Författarnas nationaliteter återspeglar temagruppens mål, att ge en översikt över läget år 1996 i fråga om den internationella forskningen om undervisning i matematik utgående från ett konstruktivistiskt synsätt. Studierna är såväl teoretiska som empiriska, och de visar på variation i fråga om forskningsintressena. Tyngdpunkten förefaller dock ligga på sociala aspekter inom den övergripande konstruktivistiska ramen. Social konstruktivism – tillsammans med andra liknande teoretiska utgångspunkter – förefaller således att ha nått en stark position världen över i fråga om forskningen i matematikdidaktik.

Beställningsadress:
Åbo Akademi, Pedagogiska fakulteten, PB 311, 65101 Vasa, Finland
Tel: +358-(0)6-3247 111 Fax: +358-(0)6-3247 302

Contents

Some Questions for Constructivists to Answer 1
Ole Björkqvist, Åbo Akademi, Finland

Constructing Mathematics, Learning and Teaching 5
Barbara Jaworski, University of Oxford, UK

Constructivism and Constructivist Teaching - Can They Co-exist? 29
Carolyn A. Maher, Rutgers University, USA

The Struggles of a "Constructivist" Curricular Innovation 43
Paola Valero, Universidad de los Andes - Bogotá, Colombia

Exploring the Socio-Constructivist Aspects of Maths Teaching: Using "Tools" in Creating a Maths Learning Culture 61
Anna Chronaki, University of Bath, UK

Study of the Constructive Approach in Mathematics Education 85
Tadao Nakahara and Masataka Koyama, Hiroshima University, Japan

Basic Imagery and Understandings for Mathematical Concepts 105
Peter Bender, Universität-GH Paderborn, Germany

Context as Construction 129
David Clarke and Sue Helme, University of Melbourne, Australia

Some Questions for Constructivists to Answer

Ole Björkqvist
Åbo Akademi, Finland

The introduction to the two sessions of Topic Group 6 at ICME 8 included some challenges for constructivism as a framework for mathematics education.

Mathematics teaching from a constructivist point of view

Constructivism need not have anything to do with teaching at all. Constructivism need not have anything to do with mathematics at all. As a scientific framework, it is concerned with the way human beings come to know certain things, and thus it is most closely related to philosophy (epistemology) and to the psychology of learning.

Nevertheless, as of 1996, constructivist views have been extremely prominent within mathematics education for more than a decade. Constructivism is still a very popular framework for research and development in mathematics education in many parts of the world. With respect to theory, you will find several varieties. Some people claim that it is a fad, which will meet the fate of other movements. It will die and make way for something else. The argument for this is that there are important questions that it seems not to have answers to. With regard to phenomena of mathematics education, its descriptive or explanatory power is impressive, but the predictive power is not comparable at all.

A constructivist framework does not imply a specific method of teaching mathematics. Yet it is not uncommon for people to regard themselves as "constructivists", meaning strongly inspired by constructivist views in their teaching. If this is to be the general description of a "constructivist teacher", the degree of inspiration may vary, which will constitute a problem in research. There will clearly exist a continuum of differences in teacher thinking and teacher behavior between the "constructivist" and the "non-constructivist" teacher. What one study describes a constructivist classroom may be a traditional classroom with just a small amount of change, as seen from another point of view. What is more, traditional classrooms are not necessarily equivalent to non-constructivist classrooms.

If we are concerned with the effects of constructivist views on mathematics education, we are faced with the challenge of specifying more clearly than previously the relationship between theory and practice. We need to clarify for ourselves the difference between direct and indirect consequences of adopting constructivist views. Having said that there are no immediate implications of constructivism with respect to methods of teaching, it is still possible for research in mathematics education to study the interpretations mathematics teachers make of constructivism, as well as any variable of interest that may be related to those interpretations.

There are, of course, also teachers who might be "defined" as constructivists in terms of classroom behavior. (This need not be due to an intentional choice by themselves.) Is it possible for research to come to an agreement about the characteristics of such behavior? The way constructivist theories in mathematics education have evolved, it seems that a constructivist teacher must be described in much more detail than as a "facilitator of learning". In fact, much research focuses on interaction and the construction of shared knowledge or meanings in the classrooms. At least from a social constructivist perspective, in a constructivist classroom there should then be some evidence of "negotiation of meaning" or similar activity. One may go on adding characteristics to achieve closer correspondence to theory.

However, there is a problem in that none of the characteristics is a necessary consequence of a constructivist framework. E.g., the phrase "negotiation of meaning" is used both by interactionists and by social constructivists. It seems that constructivism is compatible with interactionist perspectives to a very high degree. Extending the argument of non-specificity – if you want to explain the phenomena you see in mathematics classrooms, you have your choice of theories to choose from. Could it be that constructivism today is superfluous? Or is the opposite the case? Are there classroom situations where the unique best description of the phenomena is given by some variant of constructivism?

Even if there are competitive frameworks, they seem not to be as strong today as constructivism. This is particularly true, with respect to both theoretical and empirical research, for social constructivism. It seems to be a theory that resonates well with educators of today. This may be so because it appears to sum up a lot of experience and to take into account much more than previous varieties of constructivism, notably the social dimensions of knowledge and learning, the importance of which are so evident, e.g., to classroom teachers.

Constructivism as a philosophy of mathematics education may not have much predictive power, but its appeal to educators makes it a good starting point for developmental work. If it is considered primarily as a higher-level theory, under which solutions to specific problems can be found using lower-level theories, the challenge is a different one. Is it all theoretically coherent? Do the lower-level theories, which might have been chosen in an eclectic way, contradict each other or the constructivist philosophy?

Adopting the latter perspective of constructivism, our practical approach as mathematics educators (teachers and researchers) will be to avoid inconsistencies. We exclude that which directly conflicts with the philosophy of constructivism – e.g., transfer of knowledge from one person to another – but otherwise constructivist classrooms may have almost any character.

Constructivism, mathematics education and society

In many countries, effectiveness and teacher accountability are stressed as part of a national emphasis on mathematics education. A key question for constructivism is, can it be defended as a framework for better teaching, the way society sees quality in education? Is it possible to clarify the important issues and derive solutions to what society sees as the real problems just as easily from other theories?

In some countries, constructivism has been accepted very well as a philosophy of mathematics education at a national level. The guidelines for the teaching of mathematics include statements that urge you to view students as active learners, etc. However, the practical advice, regarding how to do it, is very scarce. An intentional national curriculum may be very far from what is implemented in schools. The challenge here is to accomplish large scale change. It includes aspects that are well-known from any kind of educational change – changing people, changing institutions, changing the discourse and, finally, coping with the time delay between the actions and the effects of the actions.

One of the biggest questions of all is that of assessment (Björkqvist, 1995). Nothing is thought to influence learning as much as the specific methods of assessment that are used. And if we want constructivist ideas in mathematics education to catch on in society as a whole, we need to develop methods of assessment that are reasonable to students, to their parents, to mathematicians – to everyone involved – and that do not compromise with the philosophy of constructivism.

Once again, it appears that social constructivism is best equipped to deal with the issues of mathematics education that directly concern society and its institutions. It considers, in an explicit way, interactions between the knowledge of the individual and the knowledge of a group, it emphasizes the importance of the teacher role, and variables describing social conditions and social change have a natural place in it.

Is constructivism growing old?

Well, maybe, but so are we all. (Most people prefer calling themselves mature.) In the case of constructivism in mathematics education – it is still developing new ideas and it is clearly able to cope with itself.

Reference

Björkqvist, O. (1995). Assessment in Mathematics from a Social Constructivist Viewpoint. In E. Pehkonen (Ed.), *NORMA-94 Conference. Proceedings of the Nordic Conference on Mathematics Teaching in Lahti 1994.* Helsinki: Department of Teacher Education, University of Helsinki. Research Report 141.

Constructing Mathematics, Learning and Teaching

Barbara Jaworski,
University of Oxford, UK

This paper[1] *is about the constructing of teaching for students' effective learning of mathematics, by teachers who might be seen to employ an investigative classroom approach. It draws on research involving close observation of mathematics lessons and a study of the thinking processes of selected teachers. An investigative approach was seen theoretically to be embedded within a constructivist perspective of knowledge and learning. The research sought to characterise 'investigative' teaching and to focus on issues raised for the teachers involved. The effects of teachers' critical reflection on their teaching development, and the constructive nature of the research itself were significant outcomes of the study. Effective teaching was seen to involve a critical approach to the construction of teaching.*

Investigative Teaching from a Constructivist Perspective

A study of mathematics teaching

How do mathematics teachers construct their teaching? What is the essence of such construction? How does construction of teaching relate to construction of mathematics and construction of learning? How can mathematics teaching develop as a result of this constructive process?

In the research I discuss here, I found myself addressing the above questions as a result of exploring the nature and outcomes of 'an investigative approach to the teaching of mathematics' (for brevity, to be labelled 'investigative teaching'). Investigative teaching, although not well-defined[2], was seen as an alternative to what Celia Hoyles (1988) describes as 'a transmission model of teaching and learning [mathematics] where knowledge and expertise is assumed to reside with the teacher'. Hoyles reflects on classroom constraints which obviate effective teaching:

> We know that teachers and pupils tend not to search together in a genuine and open way to uncover mathematical meaning. We know, for example, that pupils want teachers to "make it easy" or "tell them the way" and we have to recognise the powerful influences on teacher practice which almost compel an algorithmic approach. We need to find a significantly different mode of education and practice in our classrooms, new roles for teachers which they value and which they see as significant for the mathematics learning of their pupils. (p. 162)

My motivation for undertaking the research resulted from dissatisfaction with the effects of the teaching to which Hoyles refers, and from a personal excitement with the potential of investigative approaches to teaching mathematics to provide for the needs she highlights. The field work, which took place between 1985 and 1989, was designed to explore classroom manifestations of an investigative approach and the related thinking of the teachers who implemented it. There was no intention of reforming or changing practice.

The mathematics teachers who were subjects of my study were chosen because they or I considered their teaching to be in some way investigative. The fieldwork for the study involved extensive participant observation of lessons of the chosen teachers; long hours of discussion with the teachers about their thinking, their planning for lessons and their reflections on what occurred; some informal interviewing of students; some use of video-tape for stimulated response; and a small amount of questionnaire data. It drew methodologically on the research literature in classroom ethnography (eg Stubbs and Delamont, 1976; Burgess, 1985, Hammersley, 1986) and on sociological traditions in interpreting social interactions (eg Schutz, 1964; Blumer, 1966; Cicourel, 1973). At the time, there was little research of this form in mathematics education itself. Highly significant to the research outcomes was the involvement of the researcher (myself) and the extent to which this involvement coloured the research and its findings (Jaworski, 1991, 1994).

Constructivism

Investigative teaching might be seen to be motivated by a desire to develop students' conceptual frameworks in mathematics for which a metaphor of individual *construction* of understanding and its mediation through social interaction is powerful. The research took place during growth in the mathematics education community of a belief in *constructivism* as a theory of mathematical cognition and learning (eg von Glasersfeld 1983, 1984, 1990, Cobb and Steffe, 1983, Ernest 1991, Taylor and Campbell-Williams, 1993)[3]. Mathematics educators were starting to explore the implications of a constructivist perspective for the effective teaching of mathematics. Effective teaching might be conceptualised as providing an environment in which students develop a wide *relational* understanding of mathematics (Skemp, 1976) and *higher-order thinking skills* in doing mathematics (Peterson, 1988). It was part of my research to consider how constructivism might inform a study of investigative approaches in classroom teaching and to evaluate critically in what ways the teaching might be considered effective[4] .

It became important, early, to recognise that constructivism was a theory, or perspective (Ernest, 1991b), of cognition and learning, and that it was dangerous, albeit seductive, to draw inferences for teaching (Kilpatrick, 1987). In studying closely the thinking of the teachers, I was able both to recognise aspects of their thinking which I would label as constructivist [One teacher said, wryly, after reading one of my accounts "Oh, so I was a constructivist before I knew what constructivism was" (Jaworski, 1994, p 132)], and to recognise tensions in and issues arising from their teaching in accordance with a constructivist philsophy. What became most significant for the research and its outcomes was seeing the research itself and its insights into teaching development in constructivist terms. The relationship between *radical* and *social* constructivism (Ernest, 1991, Jaworski, 1994) was central to the outcomes of the research.

Research outcomes

An important objective of my research was to characterise an investigative approach to mathematics teaching through the study of the teachers involved, seeking for common features and issues. Any attempt at generalisation from ethnographic or interpretive research is problematic. Delamont and Hamilton (1984) have pointed out, however, that it is possible to 'clarify relationships, pinpoint critical processes and identify common phenomena' from which 'abstracted summaries and general concepts can be formulated, which may, upon further investigation be found to be germane to a wider variety of settings' (p. 15). This was very much the case in my research, and one of its results was to highlight teaching actions and outcomes which seemed to pertain to the classroom environments studied and might be seen as characteristic of an investigative approach to teaching mathematics (Jaworski, 1991; 1994).

For example, teachers' management of the learning situation involved fostering an ethos to support mathematical talk, collaborative working, negotiation of ideas and justification of conceptions and strategies. This resulted in students sharing responsibility for their own learning with much evidence of high-level thinking. Teachers were sensitive to students' individual propensities and needs in judging appropriate levels of cognitive demand. These characteristics demonstrated in practice recommendations from the 1982 Committee of Enquiry into the Teaching of Mathematics in Schools in England and Wales (The Cockroft Report): for example, that mathematics lessons should incorporate aspects of problem-solving, practical and investigational work, and mathematical discussion (Para 243). Hoyles (1988) suggested that such 'exhortations' were affecting, at least superficially, what was going on in UK classrooms. However, she continued:

> ... but will mathematics teaching be more effective? Encouraging pupils to talk about their methods and perceptions, to justify their strategies of exploration and proof, and to learn from each other implies a major shift in the social relations in the classroom which may not be acceptable to some teachers ... (p. 159)

It seemed that there were many respects in which the teaching I observed could be considered effective, both in terms of its embodiment of the characteristics observed and in terms of students' mathematical thinking and concept formation. This did not mean that the teaching was unproblematic. The investigative nature of the teaching approach resulted in many questions about teaching as well as raising issues and tensions for the teachers.

However, perhaps even more significant than these characteristics were the insights which the research provided into the development of mathematics teaching and its relationship to the research itself. It became clear that the teachers concerned, influenced perhaps by being the subjects of research, were actively reflecting on and developing their own teaching through tackling questions and issues. How was this related to the characteristics observed? Could it be that the major shift of which Hoyles talks is related to the reflective activity of the teachers, and that this, in its turn, is consequent on constructivist-related thinking on the part of the teachers?

Throughout the research, issues and tensions arose which guided the progress of research and substantiated or initiated theory of teaching[5] . In building a story of effective teaching – resulting from the reflective practice of teachers who act from a constructivist perspective of cognition and learning – I have to be aware of my own position and personal theories. As I worked with the teachers, my interactions with them and their classes were significant to research data. Interpretations were always through the lens of the researcher, although I sought verification by triangulation and respondent validation. Rigour lay in a reflexive accounting process where I had to examine critically my interpretations and judgments, taking into account their full situation and context. My own thinking and perspective were a part of this context.

Summary

I shall end this section with a brief summary of the position which I have presented, before going on to look at some of the research data and analysis which contributed to this position.

A number of teachers, implementing an investigative approach to the teaching of mathematics could be seen to operate effectively from a perspective based on constructivist principles of cognition and learning. Issues and tensions resulting from this operation raised many questions for the teachers to tackle. The reflective process involved in tackling these questions, leading to the teachers actively developing their own practice, could be seen as central to the effective nature of the teaching. The research took an overtly critical attitude to all interpretations and judgments.

In the next section, to support this position, I shall

- provide examples of significant episodes in the data, their analysis, and levels of interpretation;
- provide an indication of the nature of teachers' thinking and its development;

- indicate links between teachers' thinking and the mathematical learning and understanding of students;
- illustrate issues in researcher centrality and research rigour.

Research Episodes Providing Theoretical Insights into Effective Teaching

Teacher thinking in the design of teaching

The teachers with whom I worked were all guided by a mathematical curriculum which included traditional mathematical topics. For example, all teachers were required to address *Pythagoras' Theorem* with students in their classes. A major objective was that students would 'understand' or in some sense 'know' Pythagoras' Theorem. The teachers' planning process included a design of activities to bring students into a domain of thinking which would address this topic. Their subsequent evaluation involved determining how successful the activities had been in developing this understanding and knowledge. Explicitly or implicitly, the teachers had to address the meaning of understanding and knowledge.

What does it mean to *know* Pythagoras' Theorem? What are the images which underpin this knowledge? For example, we may think of a statement such as "the square of the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides", or some diagramatic form, perhaps of squares and triangles; or perhaps a deductive proof. In what ways can students be enabled to develop their own images? What would it mean for a student to have learnt Pythagoras' Theorem? Suppose *you* had to teach Pythagoras' Theorem as a topic in a secondary school classroom. How would you introduce it? What would you ask the students to do? What would you expect to be the result of your classroom activity after a number of lessons?

These questions were central to teachers' thinking in designing teaching situations. For example, as a result of such thinking, one teacher, Mike, decided to offer his class two tasks to be undertaken by groups within his class:

Triangle Lengths

Draw a triangle with a right-angle.

Measure accurately all 3 sides.

Can you find any relationship between the three lengths?

Square Sums

$$1^2 + 2^2 = 5$$

What other numbers can be made by adding square numbers together?

Investigate.

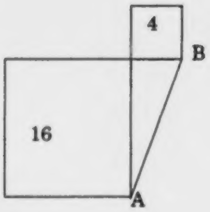
Mike hoped that by working on these two tasks, students would be in a position to appreciate the meaning of Pythagoras' Theorem[6] although it was not his expectation that they would discover the theorem as a result of the tasks. Nevertheless, he himself had clear images of what he would ultimately like his students to know. What occurred in the lesson raised many questions related to his intentions for the tasks, his expectations of students' learning and his own knowledge of mathematics and pedagogy. An analysis of episodes from this lesson may be found in Jaworski (1994, Chapter 7).

I think we must recognise, as teachers, we not only have images of the mathematics we know, we also, explicitly or implicitly, have images of what it means to know and to learn mathematics. It seems reasonable to suggest that our teaching is based fundamentally on these images. Research has suggested that teaching is influenced by teachers' own perspectives of mathematics and its learning (eg Thompson, 1984; Sanders, 1993). Lerman (1986) found that, despite differences in teachers' perspectives of mathematics, there was considerable uniformity in their teaching, perhaps indicating that their images of learning and teaching were more dominant than their images of mathematics. My research shows evidence of teachers actively questioning these influences on their teaching.

Mathematical learning: issues for teaching

Another teacher, Ben, planned a lesson on *vectors*. He expected that in working on vectors, students would need to review their work on Pythagoras which had taken place some weeks earlier. At that time, Ben had said to me that he felt one student, Luke, had not yet really understood the Pythagorean relationship. I shall refer to Luke's development of knowledge related to Pythagoras' Theorem to address some of the above issues from a practical perspective.

During the vectors lesson, I sat alongside Luke and his partner, Danny[7]. They had been asked to make up their own questions about finding the length of a vector, and to try out the questions on their partner. Luke explained to Danny what he thought they had to do. He wrote down the vector AB, as below, placed points A and B on a grid, drew the triangle around them, drew squares on two sides of the triangle as shown.



$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Luke's explanation to Danny

He wrote the square numbers in the squares, and then worked out mentally, talking aloud: "16 plus 4, that's 20; square root (pause) about 4.5". Danny seemed to follow what he had done, and each of the pair, independently, set about inventing vectors and finding lengths. Luke, in each case, drew a diagram similar to the one above, writing the square numbers into the squares. He then performed the calculation mentally and wrote down the result.

Another person in the class invented a vector with a negative number in it and asked others how to find its length. It was Luke, eventually, who said that a negative number did not make any difference – his method would still work because you just had to multiply the number by itself, and then you had a positive number as before.

It seemed to me, from my position as observer, that Luke's knowledge of Pythagoras' Theorem was at least adequate for his needs in finding lengths of vectors. There was some sense, then, in which he 'knew' Pythagoras' Theorem. This was particularly significant in the light of the teacher's comment about Luke's earlier lack of understanding. What had happened in the meantime? What learning had taken place? What growth of knowledge? And how had it happened?

Addressing such questions can help us become more knowledgeable about the pedagogy of mathematics. Teachers, constructing teaching, not only have to recognise learning and understanding in students, but have to create opportunities for their knowledge to develop, and evaluate the situations and interactions which result. The management of this learning environment is a highly complex process. Caleb Gattegno (1960) said the following:

> When we know why we do something in the classroom and what effect it has on our students, we shall be able to claim that we are contributing to the clarification of our activity as if it were a science. (Gattegno, 1960)

Luke's teacher, Ben, was aware of Luke's developing understanding. He had planned the vectors' tasks knowing that consolidation of understanding of the Pythagorean relationship was necessary. In Luke's case it seemed that his teaching had been, to some degree, effective. In our research together, Ben and I questioned the processes involved.

Reflecting on teaching and learning

It was Ben's declared aim to use *an investigative approach* to his teaching of mathematics. On a number of occasions, he referred to his teaching as being 'more *didactic* than usual'. He seemed to suggest that to be 'didactic' implied being less 'investigative'. It was in exploring what he meant by the term 'didactic' that I gained insight to important issues for Ben. This led to a recognition of links between a constructivist philosophy and classroom manifestations of an investigative approach.

I shall refer to a conversation which took place just before the vectors lesson. Ben was talking about his plans for the lesson, and seemed to be apologising because he felt it would be less investigative in spirit than he would like, or I would expect. My degree of influence here is significant for the outcomes of the research. I tried to make clear that my purpose was not to judge what I saw, but sincerely to find out as much as possible about what motivated it and its implications. However, my focus and questions, may have prompted Ben to justify his actions in terms of his perception of my expectations. There seemed to be some way in which he did not regard what he was about to do in the vectors lesson as *investigative*. He referred to it as *didactic*, and other than his normal style.

The same day, I had observed Ben take a lesson for an absent teacher involving the topic of probability. Students had experienced difficulty in using certain formulae relating to probability. This is mentioned in the conversation between Ben and myself from which Extract 1, below, is taken.

"Very didactic"

(1) Ben [The lesson will be] very didactic, I've got to say, compared to my normal style. But we'll see what comes out. There's still a way of working though, isn't there?
BJ That's something that I would like to follow up because you say it almost apologetically.
Ben Yeah, cos I /Yeah, I do. Erm.
We're back to this management of learning, aren't we?
BJ Are we?
(5) Ben Can I read what I put here? [He refers to his own written words about a concept which we called 'management of learning'] I put here, "I like to be a manager of learning as opposed to a manager of knowledge", and I suppose that's what I mean by didactic – giving the knowledge out.
BJ Mm. What does, 'giving the knowledge' mean, or imply?
Ben Sharing my knowledge with people. I'm not sure you can share knowledge. Mathematical knowledge is something you have to fit into your own mathematical model. I've told you about what I feel mathematics is?
BJ Go on.
(9) Ben I feel in my head I have a system of mathematics. I don't know what it looks like but it's there, and whenever I learn a new bit of mathematics I have to find somewhere that that fits in. It might not just fit in one place, it might actually connect up a lot of places as well. When I share things it's very difficult because I can't actually share my mathematical model or whatever you want to call it, because that's special to me. It's special to me because of my experiences. So, I suppose I'm not a giver of knowledge because I like to let people fit their knowledge into their model because only then does it make sense to them. Maybe that's why if you actually say, 'Well probability is easy. It's just this over this.', it doesn't make sense because it's got nowhere to fit. That's what I feel didactic teaching is a lot about, isn't it? Giving this knowledge, sharing your knowledge with people, which is not possible?

Extract 1 from transcript of discussion with Ben before the vectors lesson (Jaworski, 1994, p. 157)

Ben seemed to be saying that if we offer probability to students as simply a formula, "this over this" it is likely to have little meaning for students because they have no means of 'fitting' it into their experience. He seems to use the term 'fit' in a radical constructivist sense (von Glasersfeld, 1984). He speaks of knowledge in terms of personal construction. He seems to eschew any absolute sense of knowledge (*giving* knowledge is not possible). He also makes problematic the communication of knowledge (*sharing* knowledge is not possible).

Ben seemed to offer a spontaneous articulation of a radical constructivist philosophy, emphasising the individual nature of knowledge. He saw a didactic style of teaching as being inappropriate to individual construction. I questioned what I saw as being a tension for him: that of didactic teaching approach versus constructivist philosophy. The above conversation continued as follows:

"A conjecture which I agree with"

(10) BJ I'm going to push you by choosing an example. *Pythagoras* keeps popping up, and Pythagoras is something that you want all the kids in your group to know about. Now, in a sense there's some knowledge there that's referred to by the term 'Pythagoras'. And, I could pin you down even further to say what it is, you know, what *is* this thing called Pythagoras that you want them to know about?
Ben My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge.
BJ How did they come to that?
Ben Because I set up a set of activities leading in that direction.
BJ Right, now what if they'd never got to what you class as being Pythagoras? Is it important enough to pursue it in some other way if they never actually get there?
(15) Ben Yeah.
BJ What other ways are there of doing that?

He laughed and then continued.

Ben You're talking in the abstract which then becomes difficult, aren't you now? Because we're not talking about particular classes or particular groups of students etc. Because I've always found in a group of students if I've given them an activity to lead somewhere there are some students who got there. It sounds horrible that. *Came up with a conjecture which is going to be useful for the future if I got there*, yes? And then you can start sharing it because students can then relate it to their experiences.
BJ So, it's alright for them to share with each other, but not alright

for you to share with them?

Ben If I share with them I've got to be careful because I've got to share what I know *within those experiences.*

(20) BJ OK. So, if we come back to didactic teaching then, if you feel they're at a stage that you can fit – whatever it is that you want them to know about – into their experience, isn't it then alright? You know, take the probability example this morning. If you felt they'd got to a stage

(21) Ben That is nearly a definition, isn't it? That is, I suppose that's one area I'm still sorting out in my own mind. Because things like $\overrightarrow{AB}$ and *vector* is a definition. What work do you do up to that definition?

Extract 2 from transcript of discussion with Ben before the vectors lesson (Jaworski, 1994, p. 158)

The tension seemed to be between having some particular knowledge which he wanted students to gain, and the belief that he could not *give* them the knowledge. However, he encouraged students to share with each other in classroom discussion and negotiation. Ben seemed, epistemologically, to be grappling with both radical and social constructivist positions while to some extent bound by absolutism. (See Ernest, 1991a, for an account of these terms.)

The above conversation seemed to summarise his pedagogical approach – the presentation of activities through which the students could construct knowledge, and his monitoring of this construction, "My kids have made a conjecture about Pythagoras which I agree with. So, it's not my knowledge. It's their knowledge." Implicit in this is his need to know about their construction, to gain access to their construal. Students have to be able to express their thoughts in a coherent way for the teacher to make this assessment, so he has to manage the learning situation to encourage such expression – he distinguished between being a *manager of learning* and a *manager of knowledge* (statement 5). In statement 21, he referred to a *definition*. The probability example involved a definition, as did the notion of vector and its representation as $\overrightarrow{AB}$. His remark, "I'm still sorting out in my own mind" seemed to refer to the status of a definition in terms of knowledge conveyance or construction, and indeed the nature of knowledge itself. There seemed to be some sense in which you could only *give* a definition. If this is the case, what preparation needs to be done so that the student is able to fit that definition

BEST COPY AVAILABLE

meaningfully into their own experience? The tension here seems to be an example of what Edwards and Mercer (1987) call the *teacher's dilemma*. There is some concept which the teacher needs to elicit or to inculcate. However, inculcation is likely to result in lack of meaning, and eliciting of what the teacher wants may never occur (Jaworski, 1989). Statements 20 and 21 provide examples of teacher and researcher actively working on epistemological questions and the teacher overtly acknowledging his own developing thinking.

We see, therefore, a picture of students grappling with mathematics, working on tasks set by the teacher. The teacher's construction of tasks is designed to support the student's construction of mathematics. The teacher's personal philosophy of knowledge and learning is central to the construction of teaching, and the teacher actively questions his own theoretical position.

The social construction of knowledge

One characteristic of an investigative classroom approach was that students frequently worked in groups, that mathematical ideas were discussed between students and with the teacher, and that lessons involved active negotiation of meanings and understandings. I believe that it is through such overt negotiation of meaning that the teachers' dilemma is to some extent resolved. I shall address this with reference to the teacher, Mike, and his lesson on Pythagoras' Theorem.

Mike asked the class to organise itself in groups of four to work on the two tasks mentioned above. He walked around observing students' work and interacting with groups in conversations about their work and thinking. In some cases he gave advice; in others he left students with questions. Sometimes he seemed quite happy to leave them to follow their own directions. He did not compel any group to follow a particular path. Characteristic of this work was that students investigated situations which involved them in thinking mathematically in areas related to Pythagoras' Theorem. There was no expectation that anyone would discover or invent the theorem.

In the second lesson on the topic, an episode occurred which is significant to the social construction of knowledge. In one group, a boy, William, had been absent during the first lesson. The others in the group explained to him what they had been doing with the task 'Triangle Lengths'. His response was, "Oh, it's *Pythagoras* isn't it!" It appeared that he had met the result when studying at another school. They made him explain, since up to this point the name *Pythagoras* had not been mentioned, and they then checked out what he said and discovered that it fitted their data.

When Mike joined their group they were full of this experience.

It's Pythagoras isn't it!

(1)	M	Right, can you explain how far you're getting?
	S1	We've worked out the triangle lengths, haven't we William? (Laughter)
	M	How do you mean — the triangle lengths?
	S1	Em, that's it (pointing to book)
(5)	M	What can you tell me about the triangle lengths then?
	S2	Well, you take the two sides, the two right angled sides, and you square the numbers
	M	Can you tell me where this came from?
	S1	William said ...
	S2	William. We've been slogging our ideas out for days and William came in and said he knew it.
(10)	P1	That's how you do it.

Extract 3 from Pythagoras 2 (Jaworski, 1991)

The conversation continued, Mike probing to see just how much they did in fact understand and suggesting further tasks. It seemed that they understood the relationship, since they were able to explain it to Mike. At the end of the lesson I asked the group whether it had mattered that William had told them the result before they could work it out for themselves, and they appeared not to be concerned about this. Ironically they seemed more concerned about the amount of work they had put into the task, as if this would not have been necessary if someone had told them the result earlier. I was intrigued by this indication of an absolutist epistemology on the part of these students. What point did they see in their extensive exploration? What differences might there have been if the result had been presented by direct instruction?

My perception was that the group had developed a high level of common, or *taken-as-shared* meaning (Voigt, 1992). They were able to process William's contribution relative to this, and their subsequent explanation to Mike was evidence of their levels of (shared) understanding. The offering of the result came at a propitious time where individual perceptions of members of the group were concerned.

This case is just one episode where there was evidence of intersubjectivity – of meanings growing through the activity and discourse of the classroom of which the teacher was a part. Individually students were developing their own meanings, but their understanding was strongly dependent on the classroom discourse. When it came to giving an account to the teacher, various students contributed, supporting or modifying what someone else had said. Within such an atmosphere of sharing, the teacher could offer his perspective as just another voice in the social discourse. In practice, of course, the teacher's power being greater than that of individual students, the teacher had to be aware of the relative influence of his words when contributing explanation. However, there were many situations where a teacher's contribution seemed to be effective in enabling a student to make progress as evidenced by articulation of thinking. These, I felt, provided practical manifestations of Vygotsky's (1978) Zone of Proximal Development[8].

As a result of such reflection on developing understanding through interaction, negotiation and sharing of meanings, I came to see that radical constructivism alone is an incomplete, and sometimes unhelpful theory to underpin classroom learning. It leaves unanswered many questions about the construction of knowledge and how students come to know the mathematical ideas the teacher wants them to address. Through a consideration of social construction, it is possible to perceive of knowledge growing within the social discourse with the consequent development of individual understanding. Neither radical nor social constructivism deals with the status of knowledge, either in terms of where the knowledge exists or its validity (Noddings, 1990). However, it seemed that, in an atmosphere of negotiation, understandings could be articulated and challenged, so that knowledge grew in a dynamic way as part of the socially-constructed classroom. A crucial part of the activity of the mathematics teacher was the creation of an environment in which mathematical tasks could be tackled and an atmosphere conducive to sharing and negotiation could flourish.

The construction of teaching

I have made a case above for the teachers' operation to be seen to embody an explicit constructive element – the active questioning of practice and reflection on personal theory. I can present many other manifestations of teachers talking about their own learning and development (Jaworski 1991, 1994). For example, in reflecting on his first lesson on Pythagoras, Mike said, "So it has filled me with questions about that method, about what I should have done, what I should do next time. And that's what I call a success, a lesson I can go away from thinking about learning something from it." (Jaworski, 1994, p. 127) I see Mike here as actively constructing his own teaching. This involved a cycle of reflection and action in which ideas were acknowledged, practical manifestations were analysed and subsequent action was planned.

Another teacher, in writing for me about her experience in the research, said

> I found that I had to dredge up ideas from my subconscious to justify some of what I did, and discovered that much of my practices result from ancient decisions and intentional changes which have become habits through repeated application. I could still justify many of these habits on ideological grounds and would make the same decisions again, but the process of trawling my memory and asking 'Why do I let X and Y sit together?' and 'Why do I feel awkward if only boys answer my questions?' is a valuable one and I will initiate it for myself from time to time. Two thoughts struck me at about this time: How on earth can teachers be expected to function correctly on so many different levels at the same time? (No wonder I'm always so tired!) and what is going to happen when Barbara hits on something I can no longer justify? (Clare, April 1987 – in Jaworski, 1994, p. 187)

Again we see the teacher's questioning of her practice, and relating it to personal theory. I have considerable evidence of her working directly on such questions in the classroom. Her final words here speak again to my own position in this research.

From such analysis, I extracted a model for teacher development of the processes of teaching based on the reflective cycle (Jaworski 1991, 1994). I could not disregard my own position in this. The teacher-researcher relationship figured strongly in this model. Teachers suggested that the 'hard' questions which I asked were influential in enabling or provoking them to delve deeply into their knowledge and experience and challenge their own (pre)conceptions. In this respect the researcher acted as a distancing agent, enabling teachers to look more objectively at their own practice[9]. Mike later acknowledged this in responding as follows to some of my writing in this area:

> [The questions] were 'hard' because they were challenging. They were questions I thought I ought to know the answer to but hadn't clearly articulated. I felt the question was important to me. I'm also not sure your questions did 'distance' me from my practice. In fact they really took me deeper into it. That was part of the reflective process for me. To me, [this] doesn't imply distancing but separation. I can be separate but still very close. (Mike, March, 1991 – in Jaworski 1991)

I suggest that the 'reflective process' here is one of critiquing practice. It involved close scrutiny of what occurred in practice in relation to theoretical perspectives. It encouraged teachers to become more aware of their own theories, and to modify them in the light of practical experience. The process can lead to clearer knowledge and understanding of relationships between theory and practice. This was certainly true for myself as a researcher. I needed similarly to critique my practice, using whoever or whatever was available as distancing (or separating) agent – colleagues, other researchers, research writings etc. An important aspect of this critiquing of practice was a seeking for intersubjectivity with other practitioners and theorists.

I believe strongly that the construction or development of practice (teaching or research) in an overt or knowledgeable way depends on its active critiquing by the practitioner in the social domain. As a teacher, what do *I* understand by Pythagoras Theorem? What images does it create? What do I expect my students to know? How do I expect they will construct their knowledge? What experiences will it be valuable to provide? What environments will support such experience? How do the classroom manifestations of my personal images, beliefs and theories support, contradict or enhance those theories? In what ways is my knowledge of teaching developing? How does it accord with the knowledge and experience of other teachers? As a researcher, what is it that I have observed? How does this accord with the observations and perceptions of other participants? How credible is my analysis? Can I justify interpretations through their total context which includes my own theoretical perspectives and related assumptions? Can I convince others of what seems significant to me?[10]

Consequences for the development of mathematics teaching

My earlier rationalisation of generalisation from a few cases, supported by the words of Delamont and Hamilton, lends a level of objectivity to my findings which cannot go without critique. I must recognise that the findings in this study were a consequence of my interactions with the teachers concerned. This is not to say that the teachers' effective teaching of mathematics was in any way dependent on my presence. It is the interpretation of this teaching in terms of investigative approaches and reflective practice which is a result of the research. I believe these interpretations to be well justified, but they are undoubtably related to my presence as researcher.

In linking investigative teaching based on a constructivist perspective, through reflective practice and classroom action, to effective teaching, I am suggesting that we have here a model for the development of effective mathematics teaching. The research showed such a model in operation in the practice of the selected teachers. To what extent did its operation depend on the research process within which it was observed? For example, what was the contribution of the research questions to the reflective practice and subsequent action?

I am interested to pursue ways in which a model based on the critiquing of practice can be developed to underpin the development of effective mathematics teaching. Is it possible to design research programmes which involve teachers directly in forms of research or enquiry through which practice develops? A further study in this area is reported in Jaworski (1997a and b).

References

Bauersfeld, H. (1985). Contributions to a fundamental theory of mathematics learning and teaching. Invited lecture to Canadian Mathematics Education Congress, Quebec.

Bell, A.W. (1982). Teaching for combined process and content objectives. *Proceedings of the Fourth International Congress on Mathematical Education.* Boston: Birkhäuser, pp. 587-590.

Blumer, H. (1969). *Symbolic Interactionism.* Engelwood Cliffs, NJ: Prentice Hall.

Burgess, R.G. (1985). *Field methods in the study of education.* London: Falmer.

Calderhead, J. (1984). *Teachers' classroom decision making.* London:Holt, Rinehart and Winston.

Calderhead, J. (Ed.) (1987). *Exploring teachers' thinking* London: Cassell.

Cicourel A.V. (1973). *Cognitive Sociology*. London: Penguin.

Cobb P. and Steffe, L.P. (1983). The constructivist researcher as teacher and model-builder. *Journal for Research in Mathematics Education 14,* 83-94.

Cobb, P., Yackel, E. and Wood, T. (1990). Classrooms as learning environments for teachers and researchers. In R.B. Davis, C.A. Maher and N. Noddings (Eds.), *Constructivist Views on the Learning and Teaching of Mathematics.* Journal for Research in Mathematics Education, Monograph No. 4. Reston, Virginia: National Council of Teachers of Mathematics.

Committee of Enquiry into the Teaching of Mathematics in Schools, (1982). *Mathematics Counts* (The Cockcroft Report). London: HMSO.

Davis, R.B., Maher, C.A. and Noddings, N. (Eds.) (1990). *Constructivist Views on the Learning and Teaching of Mathematics.* Journal for Research in Mathematics Education, Monograph No. 4. Reston, Virginia: National Council of Teachers of Mathematics.

Delamont S. and Hamilton D.(1984). Revisiting classroom research: a cautionary tale. In S. Delamont (Ed.), *Readings on interaction in the classroom.* London: Methuen.

Elbaz, F. (1987). Teachers' knowledge of teaching: strategies for reflection. In J. Smyth (Ed.), *Educating teachers.* London: Falmer.

Elbaz, F. (1990). Knowledge and discourse: the evolution of research on teacher thinking. In C. Day, M. Pope and P. Denicolo (Eds.), *Insight into teachers' thinking and practice.* London: Falmer.

Edwards D. and Mercer N. (1987). *Common Knowledge.* London: Methuen.

Ernest, P. (1991a). *The Philosophy of Mathematics Education.* London: Falmer.

Ernest, P. (1991b). Constructivism, The Psychology of Learning, and the Nature of Mathematics: Some Critical Issues. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference for the Psychology of Mathematics Education,* Assisi, Italy: University of Genova.

Gattegno, C. (1960). What matters most? *Mathematics Teaching 12.*

Hammersley, M. (1986). *Controversies in Classroom Research.* Milton Keynes, UK: Open University Press.

Hoyles, C, (1988). From Fragmentation to Synthesis: An Integrated Approach to Research on the Teaching of Mathematics. In D.A. Grouws and T.J. Cooney (Eds.), *Perspectives on Research on Effective Mathematics Teaching (Vol 1).* Reston, Virginia: National Council of Teachers of Mathematics.

Jaworski, B. (1990). "Scaffolding" - a crutch or a support for pupils' sense-making in learning mathematics? In *Proceedings of the Fourteenth PME conference*, Mexico.

Jaworski, B. (1991). *Interpretations of a Constructivist Philosophy in Mathematics Teaching.* Unpublished PhD Thesis. Milton Keynes: Open University.

Jaworski, B. (1994). *Investigating Mathematics Teaching: A Constructivist Enquiry.* London: Falmer.

Jaworski, B. (1997a). Developing understanding of developing teaching. In V. Zack, J. Mousely and C. Breen, (Eds.), *Developing Practice: Teachers' inquiry and educational change in classrooms.* Geelong Australia: Centre for Studies in Mathematics, Science and Environmental Education, Deakin University.

Jaworski, B. (1997b). Mathematics Teacher Research: Process Practice and the Development of Teaching. *Journal of Mathematics Teacher Education*, First Issue.

Kemmis, S. (1985). Action Research and the politics of reflection. In Boud, D., Keogh, R. and Walker, D. (Eds.), *Reflection: turning experience into learning.* London: Kogan Page.

Lerman S. (1986). *Alternative views of the nature of mathematics and their possible influence on the teaching of mathematics.* Unpublished PhD Thesis. London: University of London.

Lester, F.K. (1980). Research on Mathematical Problem Solving. In R.J. Shumway (Ed.), *Research in Mathematics Education..* Reston, Virginia: National Council of Teachers of Mathematics.

Love, E. (1988). Evaluating mathematical activity. In D. Pimm (Ed.), *Mathematics, Teachers and Children.* London: Hodder and Stoughton.

Mason, J. (1990). Reflections on dialogue between theory and practice, reconciled by awareness. In F. Seeger and H. Steinbring (Eds.), *Proceedings of the fourth conference on the systematic cooperation between theory and practice in mathematics education: Overcoming the broadcast metaphor,* Institut für Didaktik der Mathematik der Universität Bielefeld. Universität Bielefeld, Germany.

Mason, J., Burton, L. and Stacey, K. (1984). *Thinking mathematically* . London: Addison-Wesley.

Kilpatrick J. (1987). What constructivism might be in mathematics education. In *PME-NA Proceedings, Montreal, September-October, 1983.*

Noddings, N. (1990). Constructivism in Mathematics Teaching. In R.B. Davis, C.A. Maher and N. Noddings (Eds.), *Constructivist Views on the Learning and Teaching of Mathematics.* Journal for Research in Mathematics Education, Monograph No. 4. Reston, Virginia: National Council of Teachers of Mathematics.

Pehkonen, E. (Ed.) (1995). NORMA -94 Conference. Proceedings of the Nordic Conference on Mathematics Teaching (NORMA -94) in Lahti 1994. Helsinki: University of Helsinki, Department of Teacher Education, Research Report 141.

Peterson, P.L. (1988). Teaching for Higher-Order Thinking in Mathematics. In D.A. Grouws et al (Eds.), *Perspectives on Research on Effective Mathematics Teaching,Volume 1.* Reston Virginia: National Council of Teachers of Mathematics.

Polya, G. (1945). *How to solve it.* New Jersey: Princeton University Press.

Romberg, T. A. and Carpenter, T. A. (1986). Research on Teaching and Learning Mathematics: Two Disciplines of Scientific Enquiry.. In M.C. Wittrock (Ed.), *Handbook of Research on Teaching, 3rd ed.* New York: Macmillan.

Sanders, S. E. (1993). Mathematics and Mentoring. In B. Jaworski and A. Watson (Eds.), *Mentoring in Mathematics Teaching.* London: Falmer.

Schön D.A. (1983). *The reflective practitioner,* London: Temple Smith.

Schön D.A. (1987). *Educating the reflective practitioner,* Oxford: Jossey-Bass.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Schutz, A. (1964). *Collected papers II: Studies in social theory.* A. Broderson (Ed.). The Hague: Nijhoff.

Shulman, L.S. (1986). Those who understand: Knowledge Growth in Teaching. *Educational Researcher, 15* (2), 4-14.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77.*

Smyth, J. (Ed.) (1987a). *Educating teachers.* London: Falmer.

Stubbs, M. and Delamont, S. (1976). *Explorations in Classroom Observation.* Chichester, UK: John Wiley.

Taylor, P. and Campbell-Williams, M. (1993). Discourse towards balanced rationality in the high school mathematics classroom: Ideas from Habermas's critical theory. In P.C.S. Taylor and A.J. Malone (Eds.), *Constructivist Interpretations of Teaching and Learning Mathematics.* Perth, Australia: Curtin University of Technology.

Taylor, P.C.S. and Malone, A.J. (Eds.) (1993). *Constructivist Interpretations of Teaching and Learning Mathematics.* Perth, Australia: Curtin University of Technology.

Thompson, A. G. (1984). The Relationship of Teachers Conceptions of Mathematics and Mathematics Teaching to Instructional Practice. *Educational Studies in Mathematics 15,* 105-127.

Voigt. J. (1992). Negotiation of Mathematical Meaning in Classroom Processes: Social Interaction and the Learning of Mathematics. Paper presented to the Theory Group and the Seventh International Congress on Mathematical Education, Quebec, Canada.

von Glasersfeld E. (1983). Learning as a Constructive Activity. In *PME-NA Proceedings, Montreal, September-October, 1983.*

von Glasersfeld E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality.* London: W.W. Naughton.

von Glasersfeld E. (1987). Learning as a Constructive Activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

von Glasersfeld, E. (1990). An Exposition of Constructivism: Why Some like it Radical. In R.B. Davis, C.A. Maher and N. Noddings (Eds.), *Constructivist Views on the Learning and Teaching of Mathematics.* Journal for Research in Mathematics Education, Monograph No. 4. Reston, Virginia: National Council of Teachers of Mathematics.

Vygotsky L.S. (1978). *Mind in Society. The development of the higher psychological processes.* London: Harvard University Press.

Notes

[1] This paper is based on a presentation to NORMA 94 - the Nordic Conference on Teaching Mathematics: 'Theory into Practice', Lahti, Finland, September 1994. See the conference proceedings (Pehkonen, 1995) for a

written version. It was subsequently modified for a presentation at ICME 1996, Seville, Spain, in the Topic Group on Constructivism. This paper is the written version of that talk.

[2] The term 'investigative' derives from a movement in the UK, starting in the 1960s, of classroom mathematical activity in which 'thinking, decisions, projects undertaken were under the control of the learner' (Love, 1987, p. 249). As part of this activity teachers encouraged pupils to undertake mathematical *investigations* involving exploration of some mathematical problem or situation, often in an open-ended fashion. In 1982, The Cockcroft Report (Committee of Inquiry into the Teaching of Mathematics in Schools) supported this practice, and more teachers began to take seriously such investigational work. However, this work often took place separately from the 'normal' mathematics teaching in the classroom which had traditionally followed a form of direct instruction similar to that described by Romberg and Carpenter (1986) in the United States. This separation mirrored the content/process debate (eg Bell, 1982) in the problem-solving movement (eg Lester, 1980) of the nineteen seventies and eighties. Normal mathematics teaching was seen to be about mathematical content, while investigations contributed to an understanding of the processes involved in doing mathematics (Polya, 1945; Mason et al, 1984; Schoenfeld, 1985). 'Investigative teaching' is a label to describe an approach to teaching mathematical content using the investigative processes characteristic of mathematical problem-solving and investigations. The approach was largely one of enquiry and the teacher's role in this was by no means clear.

[3] Often quoted are von Glasersfeld's (1983) two principles of radical constructivism:

1. knowledge is not passively received but actively built up by the cognizing subject;
2. the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality.

Taylor and Campbell-Williams (1993) extend radical constructivism to social constructivism by adding a third principle to the two of von Glasersfeld as follows:

> The third principle derives from the sociology of knowledge, and acknowledges that reality is constructed intersubjectively, that is it is socially negotiated between significant others who are able to share meanings and social perspectives of a common *lifeworld* (Berger and Luckmann, 1966). This principle acknowledges the sociocultural and socioemotional contexts of learning, highlights the central role of language in learning, and identifies the learner as an interactive co-constructor of knowledge.

[4] In parallel with, and contributory to, my own learning, other researchers and theorists in mathematics education were developing understandings of teaching and its relation to constructivist philosophy. Most significantly for this study, Davis, Maher and Noddings (1990) chart the development of

research into teaching and learning from a constructivist perspectives in the United States; Cobb, Yackel and Wood (eg 1990) report on a study of mathematics teaching developing through grades 1 to 3, based in constructivism and moving into interactionist methodologies; Bauersfeld, Voigt and colleagues in Germany have developed constructivist-related interactionist methodologies and analyses of classroom situations (eg Voigt 1992). Taylor and Malone (1993) present an international collection of accounts of constructivist-based research and theory in the domains of both radical and social constructivism.

[5] For example, as a result of categorisation of data from one teacher, a theoretical construct, the teaching triad – management of learning, sensitivity to students and mathematical challenge, emerged as characteristic of the observed teaching. This was subsequently tested against the practice and thinking of other teachers. Details can be found in Jaworski (1991 and 1994).

[6] It is not my intention to get into deep questions of philosophy or epistemology here. I recognise that the words 'meaning of Pythagoras theorem' may imply the existence of some knowledge, some given or common meaning, and that this raises questions of the status or position of such knowledge, particularly in connection with a constructivist philosophy. See von Glasersfeld, 1990, Noddings, 1990, and Bauersfeld 1985 for discussion in this area. My own working position is to see individual meanings developing as part of the social discourse of the classroom as this paper shows.

[7] The class were used to my presence. I sat in different places in different lessons and observed silently. Occasionally I asked students questions about their work. Occasionally they addressed comments to me.

[8] This is expanded in a paper on this topic presented to the TME (Theory of Mathematics Education) Group in Venice, 1991. Copies are available from the author.

[9] There is growing research in teacher thinking outside the area of mathematics education, for example Calderhead (1984, 1987), Elbaz (eg 1987, 1990), Kemmis (1985), Shulman (1986), Smyth (1987), and others. I have drawn extensively on the thinking of Schön (1983, 1987) for his concepts of reflection in and on action. Elbaz, Kemmis and Pearce and Pickard (In Smyth, 1987) recognise some form of distancing in relationships between researcher and researched.

[10] In an interpretive study, verification resides ultimately in whether the cases presented are sufficiently convincing to other mathematics educators and researchers in resonance with their own experience (eg Mason, 1990).

Constructivism and Constructivist Teaching - Can They Co-exist?

Carolyn A. Maher
Rutgers University, USA

In the last decade much attention has been given to what is meant by "constructivism" and to the implications of "constructivism" for the teaching of mathematics (Davis, Maher, and Noddings, 1990) There continues to be considerable disagreement about the notion of constructivism and what it means to teach and learn mathematics "from a constructivist perspective". Is there such a thing as "constructivist teaching" or is "constructivism" and "constructivist teaching" an oxymoron?

What is "constructivism"?

Davis (1984) has described constructivism in relation to how knowledge is represented in a person's mind. Each person constructs his/her personal knowledge representation structures by assembling particular components. The raw material for the building process comes from an individual's prior experiences (See, for example, Davis, 1984; Davis and Maher, 1990; Goldin, 1990; Noddings, 1990; von Glasersfeld, 1990). The mental building blocks for constructing mathematical ideas are made up of constituent parts that are, for each student, both personal and idiosyncratic.

If we accept that the key mental building blocks of knowledge come from an individual's prior experiences, it follows that the mental images derived from these experiences can be used by the individual to build appropriate representations of mathematical ideas. Additional buildning blocks for constructing representations come from ideas that one has already built as a result of previous experience.

According to Davis these ideas are not usually in the form of words, although words often can guide the building of a representation from *previously built mental images*. (Davis warns that we not assume that the key mental building blocks consist of words. Only rarely is this the case.) How then might words be used appropriately in guiding student construction of mathematical ideas?

What is "constructivist teaching"?

An approach to teaching that focuses on the ideas that students are building or have already built in their minds might be regarded as "constructivist teaching". An examination of teacher role from several long-term studies (See Maher and Martino, 1996; Maher, Pantozzi, Martino, Steencken, and Deming, 1996; Maher and Martino, in press, a; Maher and Martino, in press, b; Maher, Martino, and Pantozzi, 1995; Maher and Martino, 1994; Maher, Martino, and Alston, 1993; Maher and Martino, 1992a; Maher and Martino, 1992b; Martino, 1992) of how children build new ideas and extend existing ones suggests a profile of what might begin to describe "constructivist teaching". These analyses come from a decade of research in which mathematics lessons have been videotaped with three or four cameras to capture students working on problem tasks and to observe students' interactions with other classmates and the teacher. We see in the episodes a teacher/researcher listening, questioning, and responding to children in a variety of ways. As we study the process by which children are engaged in building a valid justification of their ideas, we observe new roles for what might be called "constructivist teachers".

An example

While working on the problem "Guess My Tower"[1] that was presented to a class of ten-year old students, two children (Stephanie and Milin) remembered a rule that they had conjectured eight months earlier for generating the total number of n-tall towers that could be built when selecting from plastic cubes available in two colors. Two different forms of reasoning were used as a basis for the rule. Stephanie used an argument by "cases"; Milin developed an inductive argument. A different form of reasoning established the foundation of the ideas for each child. Of particular interest in this analysis is an examination of the development of Stephanie's conceptual understanding of Milin's idea.

---

[1] The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower that is picked matches a choice, the player wins the game. The box contains all possible towers that are three cubes tall that can be built when selecting from cubes of two colors. There are four possibilities for a winning tower: (1) all cubes are exactly the same color; (2) there is only one red cube; (3) exactly two cubes are red; and (4) at least two cubes are yellow. The children were asked what choice they would make and why a particular choice would be better than any of the others. Red and yellow plastic cubes were available for the children to build towers of varying heights.

During the course of the problem exploration, Milin shared his understanding of the doubling rule with his partner, Michelle, who in turn showed Stephanie and Matt. Stephanie, however, was not successful in explaining Milin's idea to some other classmates until Matt intervened. He placed his hand on Stephanie's shoulder and directed Stephanie to "Move over". Stephanie became attentive and was quiet during his explanation which made use of precise language and careful modeling with plastic cubes. Stephanie was again asked by the teacher why the number of towers doubled when you build from 2-cubes tall to 3, to 4. Now, however, Stephanie produced a precise explanation and demonstrated with the cubes, responding to Matt's language of "adding on" and introducing the idea of "reproducing". Her elation at her own understanding was expressed in jubilation: "Yes! I knew it! I knew it ! I knew it!... I told him all along, I was right...". This signaled the resolution of her conflict and initiated a new development of understanding.

A video story

A videotape and accompanying transcript illustrate the complexity of learning and teaching from a constructivist perspective. The story is presented with six episodes that are summarized below:

Episode 1 and 2. Stephanie and Matt with Teacher

The children had an opportunity to think further about the Tower Problem and its variation when presented with a new problem, Guess My Tower. Stephanie and Matt were working as partners to find the total number of towers 4-cubes tall. Stephanie referred to a "doubling method" and predicted a total of sixteen towers 4-cubes tall. She made reference to her earlier procedure of doubling and stated with conviction a rule for calculating the total number of towers of a given height based upon that pattern. However, she could not explain why the method worked.

Earlier Stephanie explained to the teacher: "Well a couple of us figured out a theory because we used to see a pattern forming. If you multiply the last problem by two, you get the answer for the next problem". She indicated, however, that the pattern wasn't working because she and Matt only found twelve towers. She said she would go all the way back to one [cube-tall] and start over. She acknowledged that she didn't know why the rule worked and told Matt that they didn't have "*all* the answers, all the answers we can get", that is, all the possible towers. She then revised her theory, suggesting: "Maybe it wasn't two because I know it worked. Maybe it was *adding* two".

Stephanie and Matt continued to work and considered that the theory might work only for even numbers. They decided to test Stephanie's doubling idea with smaller-sized towers, 2-tall or 3-tall. Stephanie asked for blocks and the teacher suggested that Matt would find working with Stephanie on this interesting. Stephanie responded: "Yeah, because I know this works. I'm sure of it. We'll start with one and we'll multiply our way up."

Episode 1

Line Text

1 Teach: How many kinds of towers do you think there are going to be that are four tall?
2. Matt: That are four tall?
3. Teach: Yeah.
4. Matt: Um, a lot!
5. Teach: Think so?
6. Steph: There could be more than three. Was it? All right. The last. Oh, I remember the way that you could make sure how many, but I don't know.
7. Teach: How did that work?
8. Steph: It was whatever number you got from the last one you multiply by two and then you get the number of how many there will be for the next one. But I didn't get the number for this one so I'd have to go all the way back to one and start again.
9. Teach: Why do you think that worked? Remember, we kept multiplying by two and doubling the number of towers?

10. Steph: Yeah. I got all the way up to eleven.
11. Teach: Why do you think that worked?
12. Steph: I don't know. [Teacher chuckles.] Umm.
13. Teach: Do you remember why? I think someone explained it once when we were doing all that last year.

Episode 2

Line Text

14. Matt: You'd have one this big. [Gesturing with his hands, he stretches them apart as far as he can.]
15. Steph: You could fill up the Empire State Building.
16. Teach: So you think you've got all the ones that are four now?
17. Steph: I don't know because it worked. I know it's worked. I just don't know how to prove it because I'm stumped.
18. Matt: Steph! Maybe it didn't work.
19. Steph: Oh, no. No. Because I'm pretty sure it would.
20. Teach: Is this the most convincing way you could organize these in terms of, of being absolutely sure? It's easier to see for threes, but I'm wondering if there's a way that's...
21. Steph: Um...
22. Teach: A way that's, you know, where it's easier to see, especially when the towers get taller.
23. Steph: Yeah, because that's harder.
24. Teach: To know that you've got them all because like that's the problem that you're having over here, I think. Right? Cause you're not really sure. But you think there's more, right? You seem to think there's more that are four tall.
25. Steph: Yeah. I do because
26. Teach: And you're, in fact, you are predicting that there are
27. Steph: I think we goofed because I'm still sticking with my two thing. I'm, I'm convinced that that I goofed and that I messed up 'cause I know that there's something.
28. Teach: What do you think Matt? Is there another way that those could be organized so that we could be more sure that we're finding all of them?

Episodes 3 and 4. Milin and Michelle with Teacher

Milin, using cubes to build towers as he spoke, explained to the teacher and his partner, Michelle, how the towers were generated. The teacher asked Michelle if she knew what Milin did. Michelle responded that she did not. The teacher then addressed Milin: "Michelle doesn't know what you did." Milin responded, "Okay." and modified his explanation by removing cubes. He disassembled two towers 3-tall to show Michelle how he had arrived at them and displayed the 2-cube tall tower that generated them. He noted that this process did not produce duplicates. "Take these two off, okay? There's only two colors in all." Milin responded to the teacher's questioning with his explanation and his demonstration with towers. When the teacher asked Michelle if Milin's explanation made sense, she responded: "uh-huh".

Michelle then demonstrated her understanding of Milin's explanation by interrupting Milin's explanation and extending it for towers 4-cubes tall. She expressed delight at her achievement, indicating: "This is a lot simpler from the last time we explained it." Michelle replied to the teacher's questioning of why: Becauce last time, we were, we didn't do it like this. It' easier to explain when you have it like this even though, like, I think the answer is sixteen because eight times two is sixteen, and from every one you add on every one of these, you add on two."

Episode 3

Line Text

29. Teach: Do you know what he did?
30. Mich: Mm. [no]
31. Teach: Michelle doesn't know what you did.
32. Milin: Okay. Take these two off, okay? [He removes one cube from each of two towers three high leaving two towers, two high. Each has a red cube on the bottom and a yellow cube on the top.] So, there's only two colors in all. These two are the same thing as this, right? And if you put this to make it to a three [He puts a red cube on the top of one of the towers.] you put this on this, and to make this one a three you put one on this [He puts a yellow cube

on the top of the other tower.] and that wouldn't be a duplicate.

Episode 4

Line Text

33. Teach: Now, this one here [a tower two-tall with a red on top of a yellow] I'm getting a little confused. Which is the ones that came from this? [Pointing to a tower with yellow on the bottom and red on the top.]

34. Milin: This and this. [He hold up two towers, the first one has one red on top of two yellows, while the second tower has one yellow on top of two reds.]

35. Teach: Well, this doesn't make sense to me because this doesn't have a yellow one on the bottom, and a red on the next one [Teacher points to one tower with yellow on bottom and red on top]. Is that right?

36. Mich: It is confusing.

37. Teach: Does that make sense to you? This [the next tower to be built] should have a yellow [on the bottom] and a red [in middle position, like the three-high tower with red on top, red in the middle and yellow on the bottom].

38. Mich: This, this one should be like this, yellow, red, and then yellow [on top] because then we would have a yellow on a red [middle position] and a red on a yellow because we already have a red [on top].

39. Teach: Well that makes sense.

40. Mich: Then you add the two yellows [counting next the two-high towers of two yellow cubes], red on yellow and then you add a red cause there's two different colors.

41. Teach: Okay, I see that.

42. Mich: From this [tower two-high of two red cubes] you add a red and a yellow, and on this you add a red [tower two-high with red on the bottom and yellow on the top].

43. Teach: Oh, now I see how it works, okay. So, now I see, well then, then I should be able to know how many I would have if they're four high, and how to make them.

44. Milin: Yeah.
45. Mich: Oh, this is a lot simpler, from the last time we explained it.
46. Teach: Is it?
47. Mich: Yeah.
48. Teach: Why?
49. Mich: Because last time, we like, we didn't do it like this. It's easier to explain when you have it like this, even though, like I think the answer is sixteen because eight times two is sixteen, and from every one you add on every one of these, you add on two.
50. Teach: Oh.

Episode 5. Milin, Michelle, Stephanie and Matt with Teacher

A discussion between Milin and the teacher about "who else knows this" [way of generating towers] led to Milin's suggestion of Stephanie. The teacher invited Stephanie and her partner Matt to join Michelle and Milin, indicating that they shared "something really neat." Matt seemed to grasp Michelle's explanation and commented to his partner: "Steph, that's what we were doing. It's like, it's sort of like a family tree thing." Matt then joined Michelle in her explanation, concluding "And so it's a family tree."

Episode 5

Line Text

51. Milin: Go to five [the number of towers built from cubes five-high]?
52. Teach: Go to four and then go to five, and show me if it still works. Don't you think the class would like to know this?
53. Milin: Ah, let's see who else knows this?
54. Teach: Who else know this?
55. Milin: Stephanie knows this, because we found this out, remember?
56. Teach: Stephanie knows this? I don't know.
57. Milin Well, she should.
58. Teach: Well, we ask her to come over here and look.
59. Milin: She should from last year.

60. Teach: We could invite her over. She remembers this from last year? Stephanie, do you want to come over here? And Matt?

61. Milin: If she remembers it from last year because.

62. Teach: Matt, Michelle and Milin just showed me something really neat and she says that you know this, so. I don't know.

63. Milin: I don't know about Matt, but

64. Teach: Well, maybe Matt wants to know it.

65. Steph: Is it the two times the number thing?

66. Milin: Yeah.

67. Steph: That's what I was trying to explain to Matt.

68. Teach: Well, Michelle, why don't you do it, because Milin just explained it to Michelle. Let's see if Michelle knows it, okay. Milin just explained it to me, too.

69. Mich: For this one [a tower two-high that is red on the bottom and yellow on the top] you can add a red on top of it and a yellow because there's two colors, and this one [a tower two-high of yellow cubes] you can add a red and a yellow, so, it's, there's, I don't know how to explain it.

70. Matt: Steph, that's what we were doing. It's like, it's sort of like a family tree thing.

71. Teach: A tree. Yeah.

72. Steph: I knew I was right; I knew it.

73. Matt: Yeah, you got a yellow or red on top of that, and then it's like that.

74. Mich: And then it's here, too, and here, and that's how you find out.

75. Matt: And so it's a family tree.

Episode 6. Stephanie with Large Group and Teacher

A confident Stephanie using models of towers explained the reasoning behind the doubling pattern to a larger group of students as the teacher observed: "One red, okay? And I have a yellow and for each of these you can make two because all you have to do is...you can add on a red to the red and a yellow to the red...and for the yellow you can add on a red to the yellow and a yellow to the yellow, okay?" to which Michelle interjected: "So you don't have to look for duplicates." Stephanie continued, building with the cubes, playfully referring to a family tree, with parents and kids. She indicated: "You keep adding on and then here you can add the exact same pattern."

Episode 6

Line Text

76. Steph: I have one red, okay? And I have a yellow and for each of these you can make two because all you have to do is you add on, you can add on a red to the red and a yellow to the red... and for the yellow you can add on a red to the yellow and a yellow to the yellow, okay?

77. Mich: So you don't have to look for duplicates.

78. Steph: Then each one of these has two. Like okay, if this is like a family tree [The entire groups laughs aloud.], the mother, the parents, the two kids...

Discussion

Analysis of the videotape data shows the teacher listening to students and questioning them about why they thought their ideas worked (Episode 1). The teacher continues to listen and asks if another organization might be more convincing (Episode 2). In Episode 3 the teacher engages another child, Michelle, by asking if she followed Milin's explanation: "Do you know what he did?" (Line 29). In Episode 4, the teacher/researcher probes for sense-making (Lines 35-39) and tests for understanding by posing an extension of the problem. Also, the teacher/researcher listens to the student's comparison of the newly discovered solution to an earlier one (Lines 45-49). The teacher/researcher responds in Episode 5 to Milin's suggestion of extending the problem even further to towers 5-tall (Line 51) asking him to validate his conjecture (Line 52). "Show me if it still works." The teacher/researcher asks if other students would be interested in hearing about their solution (Line 52). The children respond by asking to share their solution with Stephanie and Matt (Lines 55-57). In Episode 5, the teacher/researcher has facilitated the expansion of two-member group to a group with four members. The teacher/researcher suggests that there is something that may be of interest to them that was shared by Milin (Line 62). At this time it is Michelle rather than Milin who is asked to provide the explanation (Line 68) of Milin's idea. This enables the teacher/researcher further to assess

Michelle's understanding. Matt is observed connecting Michelle's explanation of why the doubling rule works and uses a "family tree" metaphor (Line 70). Matt gives further evidence of his understanding by joining Michelle in her explanation. The teacher/researcher, aware of the justification provided by Milin, facilitates further conversations with other children who were also thinking deeply about the problem.

Implications

Young children can be seriously engaged in doing mathematics. They can be successful in listening to the ideas of others; re-examining their own ideas; rejecting ideas that do not make sense; making connections between existing ideas; and trying to convince others of the reasonableness of their thinking. Within a variety of social settings and in classrooms in which flexibility in the curriculum has made possible opportunities to continue working on problems over long periods of times, children can construct powerful mathematical ideas as teachers restrain from telling them what to do but listen carefully to their thinking.

The "constructivist teacher" encourages children to make conjectures and pursue the reasonableness of their ideas by constructing models, comparing them, developing arguments, discussing ideas, and negotiating conflicts while working on problematic situations that either have been presented to them or that they themselves have initiated and extended. Teachers, in these classrooms, facilitate discussions and probe for better understanding of student thinking through appropriate questioning that is related to students' constructions (See Martino and Maher, 1994). Indeed, instruction can be successfully guided by the questions and situations raised by students. Children's ideas can provide the stimulus for re-examination of a problem and/or exploration of a problem extension. However, willingness to defer closure on a topic until students have had time to think deeply about it might require fundamental changes in the curriculum that would permit the exploration of ideas at deeper and more abstract levels over extended periods of time (to include days, weeks, and years). In summary, classrooms that promote "constructivist teaching" might be characterized by a teacher who:

1. Provides experiences from which a student can build a powerful repertoire of mental images to draw upon for the construction of representations of mathematical ideas;
2. Assesses and estimates the ideas that a student has built by observing their activity (model building) and listening to their explanations;
3. Encourages the students to support ideas with suitable justifications and arguments;
4. Works to build a classroom culture that encourages the exchange of ideas;
5. Pulls out and calls to the attention of students differences and disagreements;
6. Facilitates the organization and reorganization of student groups to allow for the timely sharing of information and ideas;
7. Encourages student-to-student and student-to-teacher efforts to map representations and develop modes of inquiry that might disclose deeper understanding of discrepancies;
8. Provides multiple opportunities for students to talk about and represent ideas;
9. Keeps discussion open and revisits ideas over sustained periods of time;
10. Seeks opportunities for generalizations and extensions.

Finally and most importantly, the "constructivist teacher" treats children and their ideas with dignity and respect.

References

Davis, R.B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* New Jersey: Ablex Publishing.

Davis, R. B. and Maher, C. A. (1992). Constructivism and Teaching. Paper presented to the 7th International Congress on Mathematical Education, Quebec, Canada.

Davis, R. B. and Maher, C. A. (1990). What do we do when we "do mathematics"? In R. B. Davis, C. A. Maher and N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics*. Reston, VA: NCTM.

Davis, R. B. and Maher, C. A. and Noddings, N. (Eds.) (1990). *Constructivist Views on the Teaching and Learning of Mathematics,* Journal for Research in Mathematics Education, Monograph No. 4. Reston, VA: NCTM.

Goldin, G. A. (1990). Epistemology, constructivism, and discovery learning mathematics. In R. B. Davis, C. A. Maher and N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics.* Reston, VA: NCTM.

Maher, C. A. and Martino, A. (in press a). Conditions contributing to conceptual change in building the idea of mathematical justification. In *Australian Association of Mathematics Teachers Companion Volumes of Papers* from ICME 1992 Working Group, Australia.

Maher, C. A. and Martino, A. (in press b). Young children inventing methods of proof: The "Gang of Four". In P. Nesher, L. P. Steffe, P. Cobb, B. Greer and J. Goldin (Eds.), *Theories of Mathematical Learning*. New Jersey: Lawrence Erlbaum.

Maher, C. A. and Martino, A. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education 27,* (2), 194-214.

Maher, C. A., Pantozzi, R., Martino, A., Steencken, E., and Deming, L. (1996). Analyzing Students' Personal Histories: Foundations of Mathematical Ideas. Paper presented at the annual meeting of the American Educational Research Association, April, 1996, New York.

Maher, C. A., Martino, A., and Pantozzi, R. (1995). *Listening and questioning better*: A case study. In L. Meira and D. Carraher (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education.* Recife, Brazil.

Maher, C. A. and Martino, A. (1994). *Teacher questioning to stimulate justification and generalization in mathematics.*

Paper presented at the annual meeting of the American Education Research Association (AERA), New Orleans, LA.

Maher, C. A., Martino, A., and Alston, A. (1993). *Children's Construction of Mathematical Ideas.* Paper presented at the Sixteenth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), July 1993. Brisbane, Australia.

Maher, C. A. and Martino, A. (1992a). Teachers building on children's thinking. *The Arithmetic Teacher 39* (7), 32-37.

Maher, C. A. and Martino, A. (1992b) Individual thinking and the integration of the ideas of others in problem solving situations. In W. Geeslin, J. Ferrini-Mundy and K. Graham (Eds.), *Proceedings of the Sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education.* Durham, New Hampshire: University of New Hampshire.

Martino, A. M. (1992). Elementary Students' Construction of Mathematical Knowledge: Analysis By Profile. Unpublished doctoral dissertation. Rutgers University.

Martino, A. and Maher, C. A. (1991). An analysis of Stephanie's justifications of problem representations in small group interactions. In R. Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting for the North American Chapter for the Psychology of Mathematics Education.* Virginia Polytechnic University, Blacksburg, VA.

Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher and N. Noddings (Eds,), *Constructivist Views on the Teaching and Learning of Mathematics.* Reston, VA: NCTM.

von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher and N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics.* Reston, VA: NCTM.

The Struggles of a "Constructivist" Curricular Innovation

Paola Valero
Universidad de los Andes - Bogotá, Colombia

This paper presents the experience that "una empresa docente", the mathematics education research center in the Universidad de los Andes, had in developing a constructivist curricular innovation for the teaching of mathematics to first-year university, social sciences students. It depicts the organization of the main components of that curriculum and treats the series of ideological conflicts that arose during the implementation of it.

Introduction

Constructivism may be seen as a fancy word that teachers frequently mention whenever we want to show that we have a vanguard knowledge about mathematics teaching. In Colombia, a national curricular reform implemented by the National Ministry of Education tried to "sell" constructivist ideas to teachers and the programs for pre-service teacher education and even the in-service training courses show theoretically what constructivism is about. But being a "constructivist" teacher overpasses the boundaries of theoretical discourses with respect to a general epistemological approach. Being a "constructivist" teacher implies both holding a whole view that coherently connects beliefs, values and visions about the nature of mathematics, its teaching and its learning, and adopting actual teaching practices that are committed to the promotion of a better process of knowledge construction in the students. In other words, being a constructivist teacher means sharing an ideology (Ernest, 1991, p.122-123) that motivates action. And wherever ideologies appear, conflicts arise because there will always be supporters and opponents, each one trying to impose their view on the others. Succeeding in such a "battle" means facing obstacles and persuading the counterpart to, at least, listen to one's proposals.

Having this idea as a background, this paper presents the experience that "una empresa docente" had as a group of constructivist mathematics educators who designed and implemented an innovative curriculum for first-year, university, social sciences students. The main purpose of the paper is not only to depict the characteristics of such a curriculum, but also to try to analyze the series of ideological issues that the group faced during the experience. To do so, a brief reflection on the context and the basis of the innovation is presented. Then, a short description is given of the main aspects of the curricular innovation and of some of the achievements reached. Afterwards, a discussion of some of the struggles faced during the development of the experience is presented. And finally we raise some issues that we consider relevant with respect to bringing constructivism into the mathematics classroom.

Context and Foundations

The idea of developing a curriculum for social sciences students arose from the necessity of solving a problem that the Department of Mathematics of the University of Los Andes was facing. As the department in charge of the mathematical education of all the students in the institution, this department had to offer each semester a series of three courses with basic mathematical content for the students taking programs such as Political Science, Modern Languages, Anthropology, Law, Arts, Psychology and Music. The results observed with regard to the effectiveness of the curriculum implemented showed that traditional courses with a strong emphasis on mathematical content (algebra, precalculus and calculus) generated an undesired situation. These courses had high rates of failure expressed not only in the total amount of students that were able to pass each course of the series, but also in the number of students that quit the course after finishing it or the number of students who actually finished the whole series in consecutive semesters (Gómez, 1995, p. 47). Furthermore, giving these courses was more a "punishment" than a challenging teaching experience for many of the teachers.

In a context of a mainly technical university where the "important" students are those within the Faculty of Engineering, and where most of them have a positive attitude towards mathematics because it is an essential part of their curriculum, the teaching of mathematics for social sciences students was a real stone in the shoes of the Department of Mathematics. The point, then, was who did dare to try to solve the problem.

A group of four mathematics teachers in the Department of Mathematics in the Universidad de los Andes decided to take charge of the situation. In 1988, this group was the seed of "una empresa docente", which later on evolved as an autonomous research center in mathematics education within the Universidad de los Andes. By that time and until 1991, this group of people and some others who joined along the process considered themselves as "different" mathematics teachers, but did not have an awareness of mathematics education as an academic field. It is important to clarify this fact, because the initiative of formulating a new curriculum flourished from a shared, intuitive perception that the teaching of mathematics for the first and second year, social sciences students could be different from what frequently happened with engineering students. The whole alternative for solution proposed consisted of a series of three courses. The first, *Mathematics, Science and Society* (from here on MSS), presented an introduction to mathematics through its history, its philosophy and its connection to the development of science and society (Gómez, 1993b, 1995). The second, *Mathematics, Randomness and Society*, presented some basic concepts of probability and introduced descriptive statistics in the context of its role in the inquiry on the social system (Perry, 1995, 1996). And the third, *Mathematics, Statistics and Society*, presented inferential statistics as research tools for social scientists (Fernández, 1993).

This paper, therefore, is based on an *a posteriori* reconstruction of the work developed during about eight years. It also concentrates on MSS because this course clearly embodied all the issues concerning the implementation of a constructivist curriculum.

MSS: A Constructivist Proposal

In general terms, MSS was an innovative, constructivist curriculum for the teaching of mathematics for social sciences students. It was the result of the continuous, collective and reflective work of the teachers and students who were committed to it during approximately eight years. It was composed of a course, a strategy to support the teachers' practice and a series of materials to support the students´ learning. The course was taught each semester in at least six different sections or recitations, each one of which was lead by one teacher. The group of about six teachers gathered weekly in a course committee, directed by the general coordinator of the course, who was one of the researchers from "una empresa docente". These meetings supported teachers in dealing with the topics proposed for the course and with the materials that were available for the students.

In order to give a more detailed picture of MSS, let's refer to Rico's (1997) ideas about the curriculum. First of all, let's concentrate on the *classroom* or *micro-level*. In this level of the curriculum, closely connected to the relationship that teacher and students establish in order to build mathematical knowledge, there is an interaction among the *aims* pursued, the *content* treated, the *methodology* followed and the *assessment* implemented. Each one of these elements defines the expected performance that both teacher and students should have in the didactic interaction. Let's describe the main features of each of these four elements in the curricular innovation.

Aims

The course intended to introduce the students to the "world of mathematics", from an intuitive perspective and through the reflection about some historical, philosophical, scientific and social issues of mathematics. It aimed at the development of the students' capacity to handle complex, real, social situations and to construct abstract models of them in order to simplify, understand and interpret them. It also pursued the creation of a space for debate, argumentation, inquiry and problem solving among the students. And mainly, it intended to make students enjoy thinking about mathematics.

Content

With respect to the content, the course was not really a course *on* mathematics but rather a course *about* mathematics. This means that mathematical objects as such were not the center of the content, but different topics related to mathematics as a science. These topics were organized in four interconnected areas:

- *Formal systems*

This broad topic intended to explore the notion of *formal system* as one of the strongest ideas shaping mathematics. This exploration began with the notion of game and, through an adapted version of Hofstadter's MU Puzzle (Hofstadter, 1979), the students familiarized themselves with the basic components of formal systems. Then, they manipulated different kinds of formal systems such as Peano's model to generate whole numbers, Chomski's model of language, formal systems in logic and a formal system to generate fractal images, among others. Finally, these ideas were compared to the notion of social system and the use of it in the analysis developed by social scientists (Valero, 1995). The modelling of real situations was discussed as one of the purposes for the use of mathematical tools.

- *Science and society*

This topic intended to promote a reflection about the meaning of science as one of the human activities to which mathematics is closely connected. By means of studying different texts by philosophers and scientists, in both natural and social disciplines, students had the opportunity to debate the notion of *scientific method*; its implications in the development of knowledge; its recent reformulations; the difference between natural and social sciences; and the impact of scientific production on society.

- *Numbers*

The concepts of number and number system were examined through their evolution in different human cultures. The idea of infinity and recursion was also treated from various perspectives such as the mathematical with Cantor's set theory and Hilbert's paradox of the Infinite Hotel, the artistic with M.C. Escher's works, the literary with J.L. Borges' writings, and the philosophical and even religious approaches to it.

- *Puzzles*

This topic intended to present the idea of problem solving (Castro, 1995a) as one of the real-life activities that promotes the creation, development and use of mathematical concepts. Different kinds of puzzles involving some of the other three topics mentioned above were proposed to the students in order to establish a connection between the ideas discussed and the needs that mathematicians have faced when building mathematical objects.

One of the constant challenges for teachers and students was to interconnect as much as possible the topics in each of these four main areas, in order to find an articulation point that made the discussions about them as meaningful as possible. This fact implied that the content and interest of a course could vary according to the teacher's understanding of the topics and to the students' experiences, perceptions and personal and collective work. In this sense, the content of the course was not something fixed, but something changing and evolving.

Methodology

The methodology of the course could be described through the type of activities proposed to the students and the role of the instructional materials available. The activities varied greatly from individual tasks to collective tasks. Individually, the students were supposed to prepare in advance each of the sessions. They had to read some materials both from the guiding book and from papers that teachers and students suggested. In the classroom, most of the work was done in groups, either as open discussions involving the whole class, or as small groups working together. Outside the classroom the students also had to work in groups to prepare some collective work such as inquiry activities or presentations. This variety allowed the students to interact among themselves and with the teacher. As the students came from different undergraduate programs, the interaction promoted an interdisciplinarity in discussion, argumentation and critique.

With respect to the materials, students and teacher had a guide book that, rather than presenting the whole content in each of the four main topics described, introduced some basic ideas about them and formulated questions to the students and controversial issues to be discussed in class. In this sense, the book was not a traditional textbook, but mainly a guide book that offered a motivation for reflection. It was used by the students in their individual and collective work along the course. There were also some relevant readings, taken from different books, illustrating particular points of the content (Bishop, 1991; Bronowski, 1973; Polya, 1945; Sagan, 1987).

Another important tool used was a specially designed software called *mathematical didactigrams*[1]. It consisted of five different applications which presented interactive problems related to the formal systems studied in the course. The different applications included were the *MU Puzzle*, *Guess the Rule*, *Formal Systems and Language*, the *Axiomatic Method* and *Modeling a Real Situation* (Gómez, C. and Gómez, P. 1992).

The whole methodology also relied on the teacher's performance as the guide for students' work and as the proposer of challenging activities. Students, on their part, had to reflect over and analyze different topics to raise interesting issues for discussion in the classroom.

Assessment

One very first constraint of the assessment system used in the course was the necessity to fit within the general grading system imposed by the University[2]. Within its boundaries, the teacher had to use different techniques to assess the students' performance. Some basic ideas shared by the group of teachers were that assessment needed to be continuous, that teachers should be allowed to be as creative as possible in finding ways to assess, and that students should be involved in their own assessment process. According to these ideas, students had to cope with several activities such as written tests, essays, oral presentations, portfolios and production of self-made materials, among others.

---

[1] This software won the Colombian Prize for Research on Educational Software (Gómez, 1992, 1993a; Gómez, C. and Gómez, P., 1994) and its recent development (Castro, 1996, 1997) had the support of the Colombian Institute for the Development of Science and Technology, Colciencias. This software can be downloaded from http://ued.uniandes.edu.co/servidor/ued/proyectos/encurso/didactig.html/

[2] According to the standard evaluation system, students have to be quantitatively graded on a 5-point scale, 1.5 being the minimum grade, 3.0 the minimum passing grade and 5.0 the maximum grade. This restriction was an impediment to the possibility of settling a qualitative assessment system that could be more coherent with the whole proposal.

Achievements

The evaluation of MSS as a whole curricular experience that lasted for eight years is a controversial issue given the ambiguous results that could be found. The results must therefore be related to the perspective from which the evaluation is made. It was successful from the perspective of "una empresa docente" and from that of the teachers and students who grasped the meaning and coherence of the proposal and who really engaged in it. Some of the formal and informal evaluations that were carried out along the process confirm the positive results (Gómez, 1995, p. 58; Castro, 1995b, pp. 71-73). In general, the failure rates in this first course decreased and more students were able to complete the whole cycle in less time. This was considered a major improvement of the situation, in comparison to that originating the experience.

But apart from these quantitative reasons, there are several qualitative considerations to point out. For many students, the course was the opportunity to change their views about mathematics and its teaching. It was also a chance to interact with different people and to gain the ability to state a point of view and support it with a whole argument. These accomplishments could be seen in the students' perceptions:

> *This course has given me the opportunity to conceive mathematics in a different way. I was able to see what mathematics is about and what it is used for. It is much more than a set of formulas. (Castro, 1994)*

With respect to the content, two major achievements were the students' understanding of the use of mathematics as a means of modeling real situations as reported by Valero (1995), and their understanding of the notion of infinity:

> *On the one hand, most of the students are able to overcome their initial intuitive conception of infinity [as equivalent to "a lot"] and elaborate a more complex one. On the other hand, the discussions in class about the different materials studied allow the development of argumentation, abstraction and generalization abilities needed to 'think about" and "manipulate" infinity. (Valero and Castro, 1996)*

These can be seen in the quality of the reflections that several students presented in their written works and in the final exams. Students, then, were able to build more complex arguments about the topics proposed. Of course, how successful students were to interconnect the topics and to build their own ideas about them depended on the teacher's understanding of the content. In this sense, there were also positive fulfillments concerning the teachers, as expressed here:

> *The content of the course is interesting. I really think it can improve the students' reasoning processes. But I also consider that it was difficult for me because as a mathematician one has a very limited knowledge about philosophy, sciences, social sciences and general culture. Therefore, I had to learn hand by hand with the students, but that is a great experience! (Gómez, C.,1994).*

Struggles Arising from the Context

The results presented above show positive and optimistic results of the success of the constructivist experience. But we cannot deny that there were other results that highlighted a relative failure in the implementation of the proposal. These results are associated with the tensions and conflicts emanating from the connection between the *micro-level* of the curriculum and the *institutional* or *intermediate level* of it. At this level, there is a network of relationships between the educational institution and its view of how the *mathematical education* of its students should be carried out, a general *view of mathematics*, the *teachers* and the *students* (Rico, 1997). We will describe the source of the struggles that grew around MSS by means of pointing out the gap existing between the normal state of each one of these four elements and what the constructivist innovation demanded of them.

Mathematics teachers

The implementation of the curricular design highly depended on the development that each individual teacher achieved in his/her classroom. Teachers' performance was problematic and an evidence of it was the actual existence of "good" and "bad" courses and, hence, "good" and "bad" teachers. This sharp classification resulted from the students' perceptions of their own teacher. But it transcended the limits of the informal comments of the students and appeared to be reflected in the standard evaluation that the Department of Mathematics made each semester.

The main cause for this situation is the extent to which the teacher was able to involve him/herself and the students in the didactic game (Douady, 1995, pp. 65-68). This ability was difficult to gain given the teachers' conditions. Most of the teachers came from the Department of Mathematics. As expected from the kind of mathematics teaching that is supposed to take place in universities, they have a strong mathematical education and a "traditional" view of teaching. Most of them were used to explaining and lecturing about mathematical content, establishing their authority as teachers and not being questioned by students. They also lacked knowledge about social sciences.

These characteristics were deeply challenged when they were in charge of a course that needed to be taught in an unfamiliar way. First, they did not feel comfortable "teaching" the content because they did not really know it. They needed to dedicate much time to the preparation of the readings and activities, and many of them were really reluctant to do so. Second, it was hard to try to adopt a guiding position when the lecturing style was their natural way of teaching. Besides, because of their low degree of competence in the content, their model of authority based on the possession of knowledge failed. And finally, this whole situation confronted them and questioned their beliefs about mathematics, its teaching and learning. And of course, the result was adopting either a favorable attitude towards understanding the underlying basis of the proposal and transforming their teaching practice – this was the option chosen by several young teachers – or a neutral attitude that allowed them to adapt the course to make it as mathematical as possible to feel comfortable teaching it.

A very important effort made by "una empresa docente", as responsible for the general coordination of the course, was the establishment of regular and intense coordination sessions. During one hour, teachers discussed the problems they were facing in the course and shared possible solutions for them. In spite of the fact that this space became an active professional development scenario for teachers, it did not make sure that the teacher really understood the core ideas of the proposal and was able to commit him/herself to the "constructivist" process.

Students

Students built two different and almost opposite perceptions about the course. The positive opinions have already been shown. But for some other students the course was far from being an interesting, challenging, worthy experience. The explanation to this can be given not only by the perception of their teacher, but also by the series of conflicts that arose from the students themselves. Most of the students in the course were first semester students. Most of them had just graduated from high-school and their experience in maths was mainly a very traditional one. When they arrived to the course, they faced a triple conflict which impeded them from engaging in the didactic game.

First, most of them chose to study a social discipline precisely because they wanted to keep mathematics quite apart from them; but still the University forced them to join those courses. This very first conflict was solved in many cases when the students were able to realize the nature of the course they were in, or when they simply accepted that fact as part of the University rules.

Second, first semester students have a paradoxical perception of their status. On the one hand they consider themselves "big" because they already passed high-school. But, on the other hand, they are the "little" students in the university. This fact did not allow them to perceive learning as a pleasant endeavor. When they found that the course methodology suggested having fun and enjoying the intellectual pleasure of thinking about mathematics, they reacted strongly against the course and felt that it did not meet their expectations. Some of them were able to overpass this conflict when they discovered that fun was not an equivalent of easy, silly and nonsense.

And finally, students expected to find something more familiar to their view of mathematics. For most of them mathematics was a set of rules and truths that were taught by a teacher who knew everything about it, and it needed to be learned by memorizing formulas and making lots of mechanical exercises. Of course, doing this type of activities is much less time-consuming than participating actively in a series of discussions, inquiries and creative work. Therefore, in order to overpass this conflict, they really needed to be convinced that MSS was much more worthy than a normal, traditional, mathematics course. Once again, whether students were able to solve all these conflicts in positive ways depended on how the teacher was able to establish and manage the didactic contract to generate a situation in which both students and teacher were involved in the didactic game.

The institution and mathematics

Maybe the strongest struggles in the experience arose from the connection between the view that the institution, the Department of Mathematics, has about mathematics and students' mathematical education, and the micro-level of the proposal. From the very beginning of the curricular innovation there were criticisms from the teachers in the Department regarding the work that the teachers from "una empresa docente" were carrying out in this course. For some of them, excellent and very competent pure mathematicians, the course did not provide the students with real mathematical activities such as the ones that a normal mathematics course would provide. Therefore, for many of them it was a shame to have MSS among the series of mathematics courses offered by the Department.

This vision led the direction of the Department to make the decision of suppressing the course and going back to the teaching of pre-calculus for first semester, social sciences students. The evidence that supported this decision came from the standard course evaluation that the Department does each semester. This evaluation consists of a questionnaire given, at the end of the semester, to all the students taking mathematics courses. It contains a set of questions that inquire about the students' opinion of the teacher, the content of the course, the materials used, their own work and their perception of the course in comparison to some other courses taken in the University. The students have to grade from 1 – meaning the worst – to 5 – being the best – each one of the aspects considered in the questions.

As mentioned above, some students did not have a positive opinion about the course and this fact was reflected in the evaluation results. For example:

> The mean of the students' grade to the teachers of MSS was below the mean given to teachers by all the students in all the other mathematics courses. This was used to demonstrate that

> most of the teachers in charge of MSS were "bad" and had to be changed.
>
> The mean grade given to the question about the relevance of the content of MSS was below the mean for the same question in the whole Department. This finding was interpreted as a proof of the irrelevance of the content for the students' mathematical education.
>
> The mean given to the book used in MSS was significantly below the mean given to textbooks used in the rest of the courses. This was interpreted as a proof of the little pertinence that the book had and that the whole proposal had, due to the fact that the book was a product of the work of the curriculum developers.

Of course, the results and interpretations given to this evaluation should be taken with a pinch of salt. The very first remark to them is that it is not possible to compare curricular designs that are based on almost antagonist assumptions with a standard evaluation format that was created to gather information about the parameters of traditional mathematics courses. Therefore, it is not possible to compare, at least with the test proposed, the perceptions of students in Integral Calculus to the perceptions of students in the MSS course. For instance, for the items about the book of the course, one question asked the students whether the book presented theory in an organized and clear way. Integral Calculus students can clearly say that Protter and Protter's *Calculus with Analytic Geometry* does or does not, but MSS students could not say that the MSS guide book does not, because the book does not present any kind of theory and its intention is far from that!

And a second remark is that the analysis of such an evaluation, given the initial bias, could be used to support, with real evidence from the students, the view that the Department had already adopted. The conclusion of the Department, then, was to try something "new" because the intended MSS innovation did not show to be successful after eight years of experimentation.

Issues of Constructivist Experiences

One of the ideas that we have tried to explore in this paper is that, when talking about constructivism as an approach to the processes of teaching and learning of mathematics, one needs to leave apart a naive position and get aware of the constraints that adopting such an approach implies. As we tried to show, implementing a constructivist curricular design depends on the series of connections not only between the elements at each curricular level, but also between the levels themselves. From this point of view, we would like to raise two points that we consider relevant to debate.

One issue relates to the suitability of implementing large-scale constructivist experiences. Some of the presentations in this topic group reported the success and obstacles found in small-scale constructivist experiences. Much of the research in this area of mathematics education has considered focused cases of one student or some students with a teacher in the context of a classroom living real constructivist practices of mathematics teaching and learning. The general perception seems to indicate that there is a high degree of success at this level. But still, there are some other people who ask for the possibilities of engaging in such an endeavour at the level of a national curriculum. Ole Bjorkqvist's reflection precisely raises this concern. Our presentation stands at an intermediate level, the scope of a constructivist curricular innovation in a series of courses for social sciences students, in one university in Colombia. What are the real possibilities for sustaining these innovations in broader social contexts where several elements may not be in favor of the innovation?

The other issue deals with the political and ideological dimension that underlies this kind of innovations. The situation of a "constructivist" mathematics educator trying to implement a curricular design could be similar to what a Russian communist should have experienced in the late 50's if he/she would have liked to convince the Americans about why the communist regime was an alternative system to be implemented. Taking away the judgements about the goodness of the two regimes, what remains from this comparison is the idea that it is a very difficult task to intend to propagate a series of ideas and its connected practices in a social context that views the educational processes from an almost opposite point. Therefore, succeeding in such an attempt implies being aware of the political struggles taking place and of the strategies needed to convince the others about the advantages of the proposal. Maybe our ingenuity, given by our conviction of doing the right thing, let us realize this when it was too late to adopt a more "political" position.

Acknowledgments

I want to specially thank Mauricio Castro who was in charge of the project MSS during its last three years and whose comments and suggestions for the improvement of this paper were invaluable.

References

Bishop, A. J. (1991). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht: Kluwer.

Bronowski, J. (1973). *El ascenso del hombre.* México: Addison-Wesley Iberoamericana.

Castro, M. (1994). Las tareas como instrumento de evaluación didáctica. *Boletín del Club EMA. 6,* 11-13.

Castro, M. (1995a). La clase: un espacio para la resolución de problemas. In P. Gómez et al., *Aportes de "una empresa docente" a la IX CIAEM.* Bogotá: "una empresa docente". Http://ued.uniandes.edu.co/servidor/ued/libros/libroaportes/.

Castro, M. (1995b). Un curso de matemáticas para ciencias sociales. In P. Gómez et al., *Aportes de "una empresa docente" a la IX CIAEM.* Bogotá: "una empresa docente". Http://ued.uniandes.edu.co/servidor/ued/libros/libroaportes/.

Castro, M. (1996). *Didactigramas matemáticos.* Http://ued.uniandes.edu.co/servidor/ued/proyectos/encurso/didactig.html/.

Castro, M. (1997). *Didactigramas matemáticos. Manual del usuario.* Bogotá: "una empresa docente". Http://ued.uniandes.edu.co/ servidor/ued/software/manualusua.sea.hqx/

Douady, R. (1995). La ingeniería didáctica y la evolución de su relación con el conocimiento. In P. Gómez (Ed.), *Ingeniería didáctica en educación matemática. Un esquema para la investigación y la innovación en la enseñanza y el aprendizaje de las matemáticas.* México: "una empresa docente".

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: Falmer Press.

Fernández, F. et al. (1993). *Estadística y Sociedad. Conceptos y herramientas básicas del método estadístico.* Bogotá: "una empresa docente".

Gómez, C. (1994). Matebásica: experiencias de un curso de matemáticas para ciencias sociales. *Boletín del Club EMA. 5,* 3-6.

Gómez, C. and Gómez, P. (1992). Modelling a real situation. *Presentation in ICME 7.* Quebec.

Gómez, C. and Gómez, P. (1994). Didactigramas matemáticos. *Presentación en el Congreso Mundial de Informática Educativa.* Lisboa.

Gómez, P. (1992). Didactigramas. Una experiencia en informática educativa. In A. Galvis and F. Rueda (Eds.). *Memorias del Congreso Colombiano en Informática Educativa.* Bogotá: Universidad de los Andes.

Gómez, P. (1993a). Didactigramas matemáticos. In Apple Computer (Ed.), *Acceso 2.0 (CD Rom).* Cupertino: Apple Computer.

Gómez, P. (1993b). *Matebásica Mática.* Bogotá: "una empresa docente".

Gómez, P. (1995). Matemáticas, Ciencia, Sociedad. Una experiencia en innovación curricular en matemáticas. In P. Gómez et al., *Aportes de "una empresa docente" a la IX CIAEM.* Bogotá:"una empresa docente". Http://ued.uniandes.edu.co/servidor/ued/libros/libroaportes/.

Hofstadter, D. R. (1979). *Gödel, Escher, Bach: An eternal golden braid.* London: Penguin.

Perry, P. (1995). Matemáticas, Azar, Sociedad: una experiencia en la enseñanza y el aprendizaje de la estadística. En P. Gómez et al., *Aportes de "una empresa docente" a la IX CIAEM.* Bogotá: "una empresa docente". Http://ued.uniandes.edu.co/servidor/ued/libros/libroaportes/.

Perry, P. *et al.* (1996). *Matemáticas, Azar, Sociedad: Conceptos básicos de estadística.* Bogotá: "una empresa docente" y Grupo Editorial Iberoamérica.

Polya, G. (1945). *How to Solve It.* London: Open University.

Rico, L. (1997). Dimensiones y componentes de la noción de currículo. In L. Rico (Ed.), *Bases teóricas del currículo de matemáticas en educación secundaria.* Madrid: Síntesis.

Sagan, C. (1987). *Cosmos.* Barcelona: Planeta.

Valero, P. (1996). El modelaje aplicado a la comprensión de problemas sociales. In P. Gómez et al., *Aportes de "una empresa docente" a la IX CIAEM.* Bogotá: "una empresa

docente".
Http://ued.uniandes.edu.co/servidor/ued/libros/libroaportes/.

Valero, P. and Castro, M. (1996). *An Interdisciplinary View of Infinity*. Bogotá: "una empresa docente", work document.

Exploring the Socio-Constructivist Aspects of Maths Teaching: Using "Tools" in Creating a Maths Learning Culture

Anna Chronaki
University of Bath, UK

Background orientation

The aim of this paper is to explore the function of varied "tools" used in teaching such as visual resources, text and verbal interactions as seeking to establish channels in communicating mathematics amongst the pupils and the teacher. Visual contexts, textbooks and classroom talk are all conventional ways to focus the participants' attention on the "learning goal" or "the maths to be taught", and essential means in conveying and shaping mathematical meanings. But how does their communicative function operate in the normal setting of a maths class? How are these "tools" interpreted, and what meanings does the teacher intent to communicate when these tools are in use?

Findings are based on the case study of a teacher working with his 11-12 year old pupils (lower secondary school) over a series of activities addressing the teaching of geometric transformations (reflection, rotation and translation), through the exploration of artistic patterns used as decorative motifs in Roman mosaics. At an initial stage in this research, a series of four activities were created which included slides of art patterns, squared paper for constructing them, and worksheets aiming to offer tasks in exploring the features of geometric transformations. The particular motif used in these activities was the pattern of Running Pelta (see fig. 1) where the three geometric movements can be explored in various ways such as: colouring shapes on the grid, observing the movement of the shapes and constructing their own transformations. The study involved research into the ways three teachers implemented these materials, as well as their associated beliefs, views and motivations.

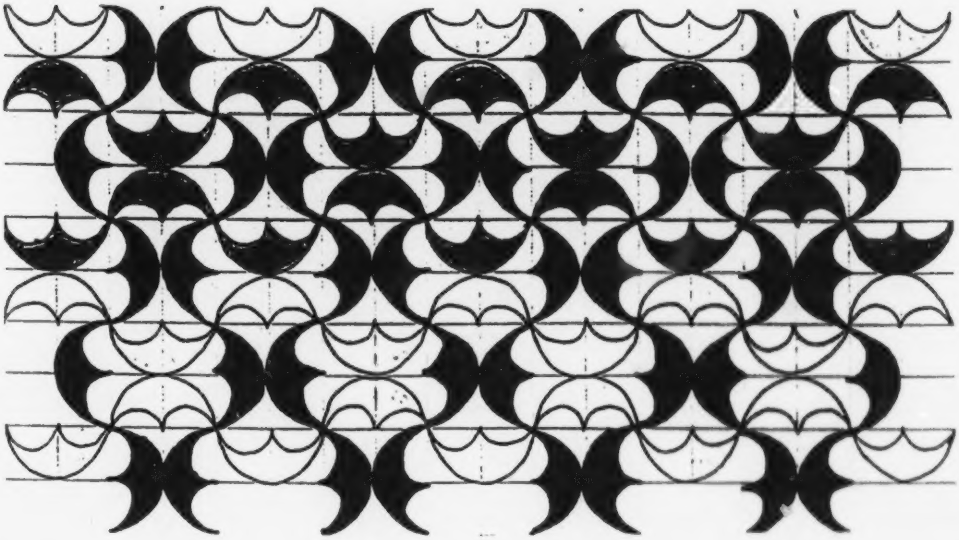


Fig. 1. The art pattern of Running Pelta

Educational ethnography is the methodological frame for this study, aiming to explore human behaviours, views, beliefs and intentions in the specific environment of the maths class (Eisenhart, 1988, LeCompte and Preissle, 1993). This frame has been used in classroom ethnographic studies (Hammersley, 1986; Delamont, 1984). Methods included participant observation, long discussions and informal interviews held with the teacher at various stages during the study. Teachers' interactions with their pupils were recorded, and their dialogues were transcribed. Also, pupils' work in the worksheets provided additional evidence about ways of working with the materials.

Data have provided a detailed description of the activities implementation in Nick's class via the analysis of specific episodes in the class and enabled interpretations about his teaching as well as about the function of the materials. Discussions with the teacher after the lessons helped in producing initial accounts and theorizing. I have produced a reflective account of my interpretations by making explicit my considerations while researching Nick's case study by presenting evidence of my thoughts and Nick's respondent validation in written accounts and by considering alternative interpretations (see Chronaki, 1996).

The shift towards social constructivism

During the last twenty years, the constructivist theory of knowing has transformed greatly our ideas about the nature and development of mathematics learning. Research outcomes based upon varied constructivist interpretations such as: "Piagetian", "trivial", "radical" and "social" constructivism have influenced us to perceive mathematical knowledge as the result of the individual's active organisation of his/her world experiences through physical engagement and reflective thinking (Steffe, 1991; Confrey, 1990; Cobb, 1994; von Glasersfeld, 1991).

Piaget's name is associated with constructivism, and he is often viewed as its main contributor of this century. Piagetian constructivism is discussed as a distinct constructivist trend and has received both vast attention and criticism. It is particularly concerned with the processes by which humans construct their knowledge of the world. It has generated the conception that learning is based on cognitive schemes about phenomena and develops through the coordination and internalisation of a person's actions on objects. Adaptation to more complex situations and assimilation of new experiences enable the already formed schemes to transform into new modified ones that express better the learning development state of the individual. Thus the learner is seen as an active organiser of his/her cognitive experiences, as the well quoted statement by Piaget indicates: "L'intelligence organise le monde en s'organisant elle-meme" (Piaget, 1937, p. 311). The implications of this theory is reflected in teaching approaches and curriculum activities which aim to provide experiences that induce cognitive conflict and hence encourage the learner to develop new schemes and organisational strategies. Teacher's role and intervention are then anticipated as promoting thought and reflection on the part of the learner.

Although Piaget has acknowledged that social interaction could play a part in promoting cognitive development through, for example, making different viewpoints available to children through discussion and group work, the focus of his research programme was how individuals make sense of the physical world through the operation of independent logical structures. The particular focus on the individual's subjective efforts in learning has become the main drawback of the theory. The fact that interactions with other cognizing subjects in a social and cultural environment have not been analysed in a substantial way as influencing cognitive development, has been recognised as an essential omission in Piaget's attempts to characterise concept formation and development. Piaget himself has acknowledged this weakness in his work. In the early 1930s, on the occasion of prefacing the Russian edition of his work, Piaget stated: "*When one works, as I have been forced to work, within a single social environment such as that of the children in Geneva, it is impossible to establish with any precision the relative roles of the individual and the social in the development of the child's thinking. In order to make this distinction, we must study children in the most varied social environments"* (quoted in Rieber and Carton, 1987, p. 90). The use of the term "social environment" can be interpreted as signifying the particular cultural and societal dynamics in a city, such as Geneva, which then influence its citizens, their activities and their developing strategies for learning in different ways as compared to citizens who live in Russia. Piaget, however, has not considered the examination of what he calls "the relative roles of the individual and the social" in learning.

Due to the above, a social constructivist perspective has emerged as an alternative, and has rapidly gained attention and interest as it claims to recognize that learning involves a process of acculturation into a social and cultural reality (Ernest, 1991; Bishop, 1988). Social constructivism has found inspiration in Vygotsky's theory of knowing that roots learning into the complexity of interacton within social environments. In his central thesis, Vygotsky argues that all intellectual development evolves from the interpersonal (interacting with others) to the intrapersonal (engaging in subjective actions and thinking) or in other words from the social to the individual. He claims that the social dimension of consciousness is primary whilst the individual is derivative and secondary (Vygotsky, 1978). This theory recognizes that understandings and meanings are constructed when individuals engage socially in talk and activities. Learning is thus a dialectic process involving persons engaged in conversation, and introduced to a cultural and symbolic world by more skilled members. Vygotsky criticising Piaget's work argued that by ignoring the influence of social interactions in shaping learning, the function of schooling and teaching cannot be adequately accounted for. He then proposed the concept of ZPD (zone of primal development) as a conceptual tool in analysing these relations. Driver et al (1995) offer some explanation of how involvement and interaction with others assists learning: *"As this happens they (the learners) "appropriate" the cultural tools through their involvement in the* *activities of the culture. A more experienced member can support a less experienced by structuring tasks, making it possible for the less experienced to perform them and to internalise the process, that is, to convert them into tools for conscious control".*

Some may argue that this perspective seeks to offer a good potential for bridging the gap left in Piagetian theory and others would be more critical about a superficial reliance on this aspiration (see Confrey, 1995, and Cobb, 1994, for a detailed analysis of the links and conflicts between the two trends as well as attempts for their coordination). However, by considering the learner's interactions with the "symbolic realities" that constitute what we come to share as mathematical culture, we can gain more understanding about the influence of teaching in classrooms in shaping learning. These "symbolic realities" can be: entities of conventional mathematical knowledge, channels for communicating mathematics, tool for developing new mathematics and organising mathematical discourses, as well as ways in which mathematical knowledge is being used and recognised in societal and cultural activities. It has been argued that the nature of these interactions and experiences can be responsible for shaping pupils' learning experiences as well as their emerging attitudes and beliefs about mathematics learning itself (Bishop, 1988; Lave, 1988).

A socio-constructivist perspective then results in viewing classrooms as forming communities and recognizes that an important way in which school students are introduced to some specific knowledge is through discourse in the context of adequate tasks and activities. Bishop refers to mathematics learning taking place as "social construction of meanings", and by being engaged in tasks and discussions students are socialised into a particular maths culture. He sees learning as a meaning making process, where the chosen mathematical activity aims to emphasize the learner's involvement with mathematics rather than the teacher's presentation of content. In this process, communication and negotiation of meanings play a significant role in the development of shared meanings amongst individuals who are albeit characterised from non-symmetrical relationships (Bauersfeld, 1988; Voigt, 1992). The discursive practices in classrooms differ substantially from the practices of professional argument and research enquiry in mathematics, which can be explained when one considers the features of the school setting (in terms of purposes and power relationships) as compared to varied institutions where mathematics is being practiced.

"Tools", "signs" and their "semiotic mediation"

Vygotsky (1978), in his attempt to characterise the development of higher psychological functions (thinking and learning), conceives learning as taking place when the individual interacts with his social and symbolic world. He tried to provide explanations by using the notions of "tools", "signs" and "mediated activity" and by employing the metaphors of "labor" and "production". Tools and signs possess a central role in his theorizing about learning and he uses them mainly to explain that learning is associated with cultural transmission and engagement with tasks and activities. He argues: *"The invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose and so on) is analogous to the invention and use of tools in one psychological respect. The sign acts as an instrument of psychological activity in a manner analogous to the role of tool in labor"* (p. 52).

Tools can embrace a variety of alternatives such as materials, tasks, social and verbal interactions, signs and varied symbols and typologies used by a community of knowledge. They can be seen as carriers of either established meanings or practices of some sort of intellectual heritage. He saw the role of tools as important in shaping learning and he once wrote: *"If one changes the tools of thinking available to a child, his mind will have a radically different structure"* (Vygotsky 1978, p. 126). This frees one from the assumption that there are enduring concepts that exist apart from the tools of inquiry and allows one to expect shifts in constructing mathematical understandings, and conceiving new mathematical ideas.

Tools and signs were viewed as playing dominant roles in human cognitive activities. Vygotsky claimed that although tools and signs cooperate synergistically and are parts of the same mediated activity, they differ in terms of their operation in orienting human behaviour and thought. He says: *"The tool's function is to serve as the conductor of human influence on the object of activity; it is externally oriented; it must lead to changes in objects. It is a means by which human external activity is aimed at mastering, and triumphing over, nature. The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is internally oriented. These activities are so different from each other that the* *nature of the means they use cannot be the same in both cases"* (p. 55).

It is important to note that the role of tools and signs is not seen merely as a direct perceptual function but mainly as mediators for constructing meaning and understanding. This has been described as "semiotic mediation" in order to denote the meanings carried by tools and signs as well as their interactive role in concept formation and teaching. It has been explained as *"..the mediation of tools, in particular signs (e.g. speech)... for the development of every human activity and the role of teaching in cognitive development"* (Bartolini-Bussi, 1991, p.3). Vygotsky extended further the mediational role of tools to psychological tools such as sign systems (language, writing, number system). He argued that language in particular is seen as not merely a tool for communication, but as a powerful channel shaping cognition: *"The specifically human capacity for language enables children to provide for auxiliary tools in the solution of difficult tasks, to overcome impulsive action, to plan a solution to a problem prior to its execution and to master their own behavior. Signs and words serve children first and foremost as a means of social contact with other people. The cognitive and communicative functions of language then become the basis of a new and superior from of activity in children, distinguishing them from animals"* (p. 28-29). Thus, tools and signs are components of some mediated activity that aims towards learning. Their function is to internalise external actions on tools into internal symbolisations through signs systems which potentially result in learning. It is argued that the central link between the thinking of the person and the influence of the social, cultural, historical and institutional setting in which the person lives is the mediational means the person uses to engage in the construction of meaning (Leont'ev, 1981)

The above concepts (tool, sign, semiotic mediation and mediated activity) assist in clarifying the role of socio-constructivist aspects of learning. Recently, many researchers have become enthusiastic about the potential of this theory and have tried to examine learning as part of teaching and as part of a wider social experience. To this end, more interpretations of the above theorizing and its fit to the reality of classrooms are essential. Below, we will see aspects of the function of such tools in Nick's teaching. Nick's meanings assigned to the role of those tools and his intentions in using them through varied interactions with his pupils will be explored.

Nick, the Teacher

Nick, the teacher, is in his early 40s and is an experienced maths teacher. He works in a secondary comprehensive school and is actively involved in the maths department activities and management. He agreed to take part in the study out of interest in his professional development. During a first stage of observing Nick teaching, detailed discussions were held concerning how he sees maths and its teaching. Nick explained that he views mathematics as a mental activity, which although based on observations is primarily a construct of intellectual efforts. He says; "*... what is mathematics and where is mathematics? Is mathematics something out there, somewhere, or is it inside pupils' minds? I go towards the view that it is inside pupils' minds. And what is going on inside here (in the mind) actually makes sense of what happens out there*". Nick also believes that the activity of doing mathematics has ultimately to be communicated between himself as a teacher and his pupils; "*The problem for me as a teacher is the words I use when they come out of my mouth, then get heard and become... maybe something else when they get to them. And that's where the class discussion comes in or the teacher pupil discussion or the pupil pupil discussion. They try to come to some kind of agreement of what these words are meaning, what we understand by them. I try to encourage pupils to see things in different ways...*". As can be seen, Nick appreciates language as an important medium for communicating mathematics, and moreover he is aware and sensitive to the fact that pupils will possibly make a different interpretation and give different meaning to the words he uses and that misunderstandings may occur. He also bases his teaching on providing a variety of experiences, such as class and small group discussions, mental imagery, and use of visual contexts. Due to the above features of his teaching, Nick was seen as a good case study for further enquiry.

A series of mediated activities: Art pattern, tasks, teacher's interaction ...

During the lesson in Nick's class, a number of representational forms of the mathematical content of symmetry were employed; slides, pictures, art patterns, specific tasks and particular questions through interaction with the teacher. All were intended to create channels for communication between the teacher and the pupils. We can conceive them as "tools", in the sense discussed in the section above, and we can roughly divide them into visual and discursive tools. As visual tools one can see the slides, the pictures and the art patterns used on worksheets, whilst the tasks and the particular verbal and social interactions with the teacher can be thought of as discursive tools, since they aim to initiate som type of dialogue with the pupils.

All suggested tasks in the worksheets introduce questions and exercises which can also act as a "mediated activity" attempting to convey the mathematical content through the worksheets to the pupils. The tasks seek more specific observations and specific actions often based on the art pattern. For example, the art pattern contains a variety of transformations which pupils could identify by colouring and describing. In all activities pupils could also use colours, rulers, protractors, tracing and squared paper and pinwheels which, according to how these could be used, could again provide "spatial experience". The tasks are based on the interplay of pupils' engagement in observing, colouring, making and at the same time describing what has been observed, coloured and/or made (see appendix 1 for a full description of all activities).

The function and use of the visual context

The use of slides as a tool aims to provide a real life context where the mathematical notion of geometric transformations can be observed and located. Consequently, the art pattern of Running Pelta acts as the backgrund that introduces the context for making detailed, systematic and organised observations. Art patterns are designs of artistic work that have a geometric nature. They are also known as geometric patterns and they are created by the repetition of a unit (which is often a smaller pattern or a single figure, usually called motif) on a limited space (which is the grid of a square or a parallelogram, see fig. 1). This unit can be imagined as a "building block", since the art pattern is constructed upon it. The geometric nature of the art patterns is due to the fact that the building block repeats itself on the grid and often follows rules of geometric transformations, such as reflection, translation and rotation. As Field (1988) explains: *"These geometric patterns radiate symmetry and order.... The patterns are built up from simple elements which seem to "grow" and develop in an almost organic or living way"* (p. 3). It is introduced to pupils through picture and tasks in the worksheets and serves as a context mainly for exploring and learning the features of the geometric transformations but also fostering the attitude that maths is related to other non-mathematical areas, such as art.

As a result, the mediating role of the slides and the art pattern as intended through the tasks in the worksheets can be described as twofold:

- providing a visual context for exploring and learning the features of geometric transformations;
- fostering the view that mathematics can be appreciated in artistic activities.

The function of the art pattern, as revealed through implementing the activities in Nick's class (discussions with the teacher, observation in lessons and examining pupils work) can be described as follows:

- The art pattern provides a visual context for exploring geometric transformations.
- It can influence the use of metaphoric language.
- It does not cover all possible examples of geometric transformations.
- It can create "false perception" about particular aspects of geometric transformations.

Pupils in Nick's class liked watching the slides and the pictures of the various artefacts projected and the were motivated in colouring shapes on the grid of Running Pelta as well as constructing their own individual art patterns. An examination of pupils' worksheets and close analysis of particular episodes in the lessons have shown that some pupils faced difficulties in making descriptions of the movements observed on the art pattern's grid. A coping strategy for many was the use of metaphoric language and informal expressions. For example, pupils refer to the movement of shapes as "travelling", and they measure distances on the grid in terms of the Pelta or Heart shapes. They would say; "Peltas are travelling same distances here", and "These lines are two Hearts apart".

It was also noticed that the art pattern does not cover all possible examples of geometric transformations, and that pupils had difficulties in perceiving the detailed features of different translations in the art pattern. For example, in one lesson episode, two pupils, having coloured some translations similar to the one given, "see" the same translation all over the grid of the art pattern, although differences amongst the coloured translations exist in terms of direction and distances between the shapes. The pupils' difficulty can possibly be attributed to a limitation of the art pattern's structure in providing "clear" representations of translations. It may also be due to these two pupils' lack of a translation image, which could help them to see different translations in the art pattern. Both reasons may prevent pupils from abstracting the mathematical features of the movement and as a result pupils from an incomplete image of the movement. Through the above, the visual context of the art pattern can be seen as having both benefits and drawbacks in terms of its function in developing pupils' learning of geometric transformations. What is of further interest, however, is Nick's mediating role between these particular tools and pupils' actions in constructing meanings from the use of these tools. Evidence through studying Nick's teaching shows that he uses the visual context in at least two different ways.

- A: Working with pupils *on* the visual context
- B: Taking the pupils *beyond* the visual context.

A: Working with pupils *on* the visual context: The visual context of the art pattern is a basis for making observations about the shapes. These observations can then generate discussion and work on specific tasks such as measuring, checking, and colouring of geometric transformations. The instructional intention through these processes of observation, discussion and task activity is to assist pupils in constructing mathematical meanings. Examples can be seen below, in certain extracts of episodes (see Chronaki, 1996, for a full description).

Example a (ibid. p. 15):

5. T: (the teacher repeats, pointing to the translations)
6. Horizontally, vertically and diagonally? What do you mean by vertically?
7. P1: that...that...(points to the pattern)
8. T: So, this is horizontal, this is vertical and that is diagonal. And are they same translations?
9. P2: No.
10. T: Why now no?
11. P2: Because the distances between them are not the same.

Example b (ibid. p. 26):

1. P: ....(the pupil colours a translation on the art pattern)
2. T: What is the distance between the shapes in this one?

(Points to a translation).
3. What is the distance between this one and this one?
(Points to two shapes in the translation).
4. P: The same.

Example c (ibid. pp. 30-31):

5. T: What pattern? How does this one go to here?
6. P: It turns round.
7. T: Right, it turns round. About where does it turn?
8. P: It goes from there on the top to there.
9. T: So, where is the turning point?
10. P: There.

In these examples, we see instances were the teacher asks the pupils to use the grid of the art pattern as a basis for careful observations. Nick asks them what they see, and then asks them to systematise and organise what they see. He questions them and he seeks further explanations of the already made observations such as; "what do you mean by vertically?", "why now no?", "what is the distance?", "what is the turning point?". These questions aim for clarifications of what the pupils really think, and justification of what they say. But they aim to lead pupils from the external observations made on the "tools" (slides, pictures, art pattern, tasks) towards some internal realization of the features of geometric transformations. It can be argued that Nick aims, through his mediation, to transform the tool of the art pattern into a sign of geometric transformations. In other words, he intends to enable them to think about the mathematical content through using the tools provided in the visual context.

B: Taking the pupils *beyond* the visual context: Nick was observed at various points introducing alternative visual contexts, or asking pupils to use blank sheets or squared paper. Instances for using those were for example, when the teacher had realised that pupils had a conceptual difficulty or that their understanding was strictly confined to the art pattern they were not able to relate their observations to different contexts. Nick's attempts were aimed at enabling pupils to make generalisations and abstractions about the features of geometric transformations on the grid of the art pattern, and to supply a richer experience of understanding the topic.

The reasons why Nick uses examples beyond the art pattern are varied and embrace at least one the following: first, the given art pattern wasn't a successful context for challenging the thinking of particular pupils; second, the teacher realised that an alternative example was needed to clarify some conceptual difficulties, and third, the teacher realised that a strong example was required to emphasize some already made observations.

The visual contexts used by Nick were pencils as embodiments for representing the movement of shapes, tracing paper for additional drawings, and use of measuring instruments for constructions. In one episode during the last lesson, Nick asks all pupils to stop what they are doing and watch him. He takes a couple of chairs and lays them out in forming a translation. Then he asks them what they would do to place another chair in the translation pattern. He asks: *"What would you do next to know exactly where to put this one?"*. In this episode we see the teacher using the chairs found in the class, in the realm of a whole class presentation and public discussion. The chairs act as an example beyond those already used in the worksheets art pattern of Running Pelta. As can be seen in the full description of the episode (see appendix 2) Natalie and Max, the two pupils who actually perform, volunteer to act out their own methods in creating translations using the chairs. The rest of the class observes what is happening.

The question here concerns whether the teacher had any special reasons for taking this action and how he relates this incident to the teaching of translation and the use of the art pattern. Hence the discussion: *"AC: What was the reason for using the chairs? NICK: Comes back that the chairs is just a vehicle. It is just a device to actually (...inaudible) what are they are doing. You can use the chairs to articulate what pictures they have in their head..."*. At this point, it seemed to me that the teacher was using the chairs context in a way possibly similar to the art pattern. Nick explains: *"The chair is a 3-D shape, that is moving by same distances in the translations. Max and Natalie then could put the third chair, the fourth, any chair to form the translation. And you can move 2-D shape, 3-D shape.... I want to go back and have a control... If you take this as a specific movement (he means the translation), and that is a unique movement to those 6 shapes (he means that these 6 different shapes are translated in same way). To know what a translation is you need to generalise that for any shape. That you can do it with chairs to generalise and take them off the context of Peltas"*. From the above it can be argued that the teacher uses the chairs as another shape and as a different example of a physical translation, aiming to reinforce the pupils' experience of the concept, and to help them generalise. The teacher says: *"You can use the chairs to articulate what pictures they have in their head."* and *"To know what a translation is you need to generalise that for any shape"*. In order to do this the teacher here finds it necessary to provide a new experience, by taking them outside of the context of the art pattern.

He talks about his use of these alternative examples: *"NICK:.... I felt happy when we went to the blank sheet of paper, which took a shape which was then moved, with the named specified transformation. So that seemed to be going beyond, seemed to be taking the structure away from the art pattern, leaving a shape that then can be moved in 2D space. That came towards the end didn't it. I think that's quite an important feature. That end bit needs to be there.... My view would be that you start off taking the strong shapes within the pattern, the source, and they're going to be moved off into different shapes and different sorts of shapes. Because then that would give me a hold to know that they can apply it to anything".*

Nick, himself, sees the importance of using plural experiences as a form of decontextualising the pupils' experiences from the one given example. This seems to make him feel more assured and secure that pupils will be helped to make generalisations about the mathematical content. Nick is aware, however, that the process of generalisation and abstraction is complex, but he feels that as teacher he needs to work in this direction. The provision of multiple visual contexts is then for him a strategy, and as he mentions in his last statement; *"... that would give me a hold to know that they can apply it to anything".*

Nick's attempts to create a maths culture

The visual context created by the slides, the art pattern described earlier is as a tool for working on the three geometric transformations. This tool is partly introduced through the pictorial aspects (slides, pictures and the art pattern itself), the story of Running Pelta and also the tasks in the activities. Pupils can create meanings and develop understandings through interaction with the materials. The teacher, though, plays a guiding role in delivering the materials, managing and organising the class, giving instructions about modes of working, emphasising some tasks rather than others, omitting worksheets or introducing new activities. Moreover, his role is most apparent in his interactions with pupils and the conversations he has with them about the mathematical topic being taught. Nick's interaction was responsive to cues coming from pupils. During the lesson he was going around the tables where the pupils were working in groups, observing what pupils were doing. Sometimes they would call him for help, and at other times he would move asking them about their progress.

The teacher's interaction with pupils acts as another "tool", which this time serves to re-direct pupils' attention towards specific questions and to remedy misunderstandings, to fill incomplete images but also to work on pupils' attitudes towards their mathematics learning. Describing the nature of the maths culture that Nick tries to create, two main aspects seem essential. First, the aspect of his mediating role whilst working with pupils on mathematics, and second, the fact that he does not isolate this work from wider pedagogical issues of pupils' learning.

Working with pupils on the mathematics

- asks specific questions about mathematics
- bases the questioning on pupils' own experiences and input
- takes into account pupils' interpretations
- builds on pupils' informal and metaphoric language
- organises and structures pupils' observations
- focuses their attention on features of geometric transformations

Considering wider pedagogical aspects

- challenges pupils' answers and mathematical understandings
- remedies misunderstandings and incomplete images of the mathematical content
- considers and values alternative methods whilst working with specific tasks
- shares opportunities for learning publicly, in groups and individually
- strives for creating positive attitudes for mathematics learning

The above aspects highlight Nick's attempts to create a classroom culture that respects the sharing and communication of mathematics, as well as values the promotion and challenging of pupils' mathematical understandings. We have already seen manifestations of the ways Nick works with pupils with particular reference to the use of the visual context. Nick's consideration of the wider pedagogical aspects can also be seen in a number of episodes of his teaching. For the purposes of this paper, reference will be given to the episode where Nick used chairs as an example beyond the art pattern (see appendix 2 and the earlier description).

As already explained, the teacher, having noticed that pupils were using two different methods for creating the period of translation, used the chair example to highlight publicly these differences. Asking Nick about his drive in making these two methods public, we can possibly explore more his intention, values and the significance he put on this action. Reflecting upon this episode in a discussion after it occurred, the teacher explains his rational: *"AC: What was your drive for using the chairs and making it public in the class? NICK: It seemed to me, that they both got methods for describing translations and to identify them. Both methods seem to work. Therefore, recognising that both methods work (... inaudible). At the end we had to go with Max's method ... because that gets you into the period of translation. But Natalie's method is also valid in describing a translation, work out that there is a translation. But it won't be able to get them through the period of translation. So, I try to get equal value between these two things, to identify differences and similarities between them. AC: At that point, were you thinking of how to teach translation? NICK: It seems to me, what I was trying to do is to say to Natalie that her method is a good method to answer the question and Max's method is a good method, too ... because Natalie may think "I've done it wrong". I was hoping that she goes away thinking "I've got a valid method of doing it. It is just by accident or by somebody's statement earlier that my way isn't the way to describe the period of translation ..."*

It is important here to note the intentions of the teacher. Nick tries to co-ordinate at least two aspects of his interactions in the above episode, those oriented toward developing pupils' learning of mathematics and those aiming to socialise them in a culture of maths teaching. Nick is quite sensitive to both. On the one hand he wants them to create an understanding of "the period of translation" and to distinguish between different methods for their construction. At the same time he also wants them to appreciate and respect the value of both methods used by their classmates in constructing translations. This is related to his concern with Natalie's possible feeling of failure if her method was rejected without discussion. The teacher says: *"I was hoping that she goes away thinking "I've got a valid method of doing it. It is just by accident or by somebody's statement earlier that my way isn't the way to describe the period of translation".*

Nick can be seen as a mediator of a maths culture that values mathematics as a "social construction". He is aware that Natalie's method is not compatible with the conventional way of constructing "the period of translation" as known in the mathematics community. But for Nick it is essential to make Natalie feel that she has some valuable method which, even though it is not accepted as "mathematically correct", is still valid and useful in helping her to solve the particular "chair problem". Even further, Nick challenges the status of the conventional mathematics knowledge as authority. By saying *"It is just by accident or by somebody's statement earlier that my way isn't the way to describe the period of translation"*, shows not only his respect of individual pupils' understandings but also his awareness that maths knowledge is a social and political construction. Within the lessons, however, Nick did not discuss openly this view but instead tried, through relevant tasks (as the episode on chairs), to negotiate with pupils the conventional mathematical language and methods used in the discipline of mathematics. Nick's role comes close to Leont'ev's (1981) description where the teacher's role is characterised as that of mediating between students' personal meanings and culturally established mathematical meanings of wider society.

Conclusion

The view that mathematics can be socially constructed via the use of symbolic tools and the teacher's mediating role is central to this paper. Ways in which the teacher introduces pupils to the cultural tools of mathematics through the use of materials, tasks, and dialectic interactions have been described (visual and discursive tools). It has been argued that the function and role of tools is multiple and depends largely on how the teacher intends to operate and use them in close interactions with the pupils. The teacher's mediating role becomes apparent through the classroom discourse. At least two important features can characterise his role. First, he listens and observes pupils attempting to create images of their state of understandings. Second, he introduces new ideas, viewpoints, ways of working and attitudes for working with mathematics which can also be seen as "tools" which seem to create entrances into a community for maths learning.

The particular role of the visual context as seen through the examples discussed above highlights that *"... the basic analogy between sign and tool rests on the mediating function that characterises each of them"* (Vygotsky, 1978, p. 54). It was noted that the slides, the pictures, the art pattern, the tasks and the alternative examples provided by the teacher were initial "tools" (or sources as called by the teacher). The tools differ in the ways they are perceived (verbally, visually or both) and the ways they are intended to be used (either as a stimulus at an initial stage through reading the worksheets and observing the pictures or through interacting with the teacher). In short, they can be a stimulus for observation, they can trigger mathematical thinking and they can check, challenge, improve and open communicative actions for mathematical meanings.

Their transformation from tools to signs of a maths culture was based upon the use made either by the pupils or the teacher. As Vygotsky (1978) has claimed: *"I should only like to note that neither can, under any circumstance, be considered isomorphic with respect to the functions they perform, nor can they be seen as fully exhausting the concept of mediated activity. A host of other mediated activities can be named; cognitive acitivity is not limited to the use of tools or signs"* (p. 54-55).

References

Bauersfeld, H. (1988). Interaction, construction and knowledge. Alternative perspectives for mathematics education. In T. Cooney and D. Grouws (Eds.), *Effective mathematics teaching*. Reston, VA: National Council of Teachers of Mathematics and Lawrence Erlbaum Associates.

Bartolini-Bussi, M. (1991). Social Interaction and Mathematical Knowledge. In F. Furinghetti (Ed.), *Proceedings of the 15th PME Conference. Vol. I.* Assisi, Italy.

Bishop, A, J. (1988). *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. Dordrecht, Netherlands: Kluwer.

Chronaki, A. (1996). *Case Study on Nick's Teaching*. Internal Report. School of Education, University of Bath.

Confrey, J. (1990). What Constructivism Implies for Teaching. In R. Davis, C. Maher and N. Noddings (Eds.), *Constructivist Views on the Teaching and Learning of Mathematics*. Journal for Research in Mathematics Education, Monograph No. 4.

Confrey, J. (1995). A Theory of Intellectual Development. *For the Learning of Mathematics 15* (1), 38-48; (2), 36-48; (3) 2-9.

Cobb, P. (1994). Where is the Mind? Constructivist and Sociocultural Perspectives on Mathematical Development. *Educational Researcher 23* (7), 13-20.

Delamont, S. (Ed.), (1984). Readings on Interaction in the Classroom. London: Methuen.

Eisenhart, M. (1988). "The Ethnographic Research Tradition and Mathematics Education Research". *Journal for Research in Mathematics Education 19* (2), 99-114.

Ernest, P. (1991). *The Philosophy of Mathematics Education*. London: The Falmer Press.

Hammersley, M. (1986). *Controversies in Classroom Research*. Milton Keynes: The Open University Press.

Lave, J. (1988). *Cognition in Practice: Mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.

Leont'ev, A. N. (1981). The problem of activity in psychology. In J.V. Wertsch (Ed.), *The Concept of Activity in Soviet Psychology*. Armonk, NY: Sharpe.

Lerman, S. (1992). The Function of Language in Radical Constructivism: A. Vygotskian Approach. Paper presented at the 16th PME Conference.

LeCompte, M. and Preissle, J. (1993). *Ethnography and Qualitative Design in Educational Research*. London: Academic Press.

Piaget, J. (1937). *La construction de reel chez l'enfant*. Neuchatel, France: Delachau et Niestle.

Rieber, R. and Carton, A. (1987). *The Collected Works of L. S. Vygotsky, Vol. 1, Problems of General Psychology*. New York and London: Plenum Press.

Steffe, L. (1991). The Constructivist Teaching Experiment. Illustrations and Implications. In E. von Glasersfield (Ed.), *Radical Constructivism in Mathematics Education*. Holland: Reidel.

Voigt, J. (1992). *Negotiation of Mathematical Meaning in Classroom Processes. Social Interaction and the Learning of Mathematics*. Paper presented at the International Congress on Mathematical Education, Quebec City.

von Glasersfeld, E. (Ed.) (1991). *Radical Constructivism in Mathematics Education*. Holland: Reidel.

Vygotsky, L. (1978). *Mind in Society. The Development of Higher Psychological Processes*. Edited by M. Cole, V. John-Steiner, S. Scribner and E. Souberman. Harvard University Press.

Appendix 1. Tasks in the worksheets

In the first activity, a familiarisation with the structure of the art pattern was aimed at, and attention was called to using language to describe orientation and position and construction of the pattern. The tasks included slides presentation (a variety of slides on tessellated art patterns as can be found in several cultural activities – paving, weaving, decorating). Then a series of worksheets (A, B and C) was suggested. Worksheet A was about exploring and describing the art pattern of Running Pelta by looking at different orientations and directions of shapes and the way of construction. Worksheet B was looking for smaller patterns in the grid of the Running Pelta. And Worksheet C was on pupils constructing and describing their own patterns out of the Pelta shape.

The second activity concerned the exploration of translation. The worksheets involved are: Worksheet A, where examples of translations found in real life are given; Worksheet B, where examples of translations in the Running Pelta are given and students are asked to describe the movement; Worksheet C, the analysis of a translation in terms of the relation of two neighbouring shapes' distance (the period of translation); Worksheet D, where students are asked to compare different examples of translations; Worksheet E, where students are asked to construct translations, and worksheet F, where students are asked to reflect on their work in previous tasks and to judge what are the important aspects of a translation.

The third activity is about exploring reflection. It includes the following worksheets: Worksheet A provides examples of reflections from real life. Worksheet B gives examples of reflections in the Running Pelta and students are asked to describe them. Worksheet C concerns the analysis of a given reflection. Worksheet D asks for comparisons between different reflections, and in Worksheet E students are asked to construct reflections about given mirror lines. Finally, worksheet F asks pupils to consider the tasks done in previous worksheets and to identify the important aspects of a reflection.

Finally, the fourth activity explores some essential aspects of rotations. It consists of the following worksheets: Worksheet A, where examples of rotations from real life are given; Worksheet B, where rotations in the Running Pelta are given and students are asked to make their descriptions; Worksheet C, where a 90 degree rotation is analysed in terms of the distances of the points from the centre of rotation; Worksheet D, providing 180 and 90 degree rotations for which students are asked to find the centre of rotation; Worksheet E, asking students to make comparisons between a variety of examples; Worksheet F, asking them to construct some rotations; Worksheet G, where students are asked to consider the work done in previous sessions in order to identify some important aspects of rotation.

An examination of pupils' worksheets (and also using supplementary sources of evidence; field notes, talking with the teacher and transcriptions of particular episodes) has shown that some pupils were motivated in making their own patterns and colouring shapes. They also faced difficulties in making descriptions and constructions of geometric transformations, and also in understanding and using new mathematical words. They often use their own metaphorical expressions.

Appendix 2. Episode: Using the chairs

Whilst working in the second activity (second lesson), the teacher noticed that pupils were using two different interpretations of the period of translation and its relation to the distance between two shapes. In the worksheets, the period of translation was introduced by encouraging them to compare the distances between two neighbouring shapes. Nevertheless, the pupils seem to have a different conception of what is a distance and how to measure it. Some of them were measuring from point to point, and some others were measuring the gap between the two shapes.

These two methods both seemed quite successful in constructing the next shape in a translation, but not in accounting for and measuring the period of translation. In the fourth lesson, the teacher decides to make these methods "public". The lesson was about summarising work on all three transformations (translation, reflection, rotation) and the pupils were working on a task where they had to measure the period of translation for two different pairs of Peltas. In the following episode the aim is to compare different ways of constructing translations and to clarify the concept of period. After the pupils had finished their measurements, the teacher used some chairs he found in the room, asking them to construct translations. We can follow what happened:

110: T: Can you put pens down please, and stop? Put pens down. Right.
If I take this chair and place it here. And I take another chair... And I turn this chair and place it here. And I tell you that this is a translation. I want you to think of what you do to make this chair come next. Can you concentrate please! To place one chair here, I put another chair here. Now, I call this a translation. What would you do next to know exactly where to put this one? Does everybody understand the question?

115: Ps: Yes.....

T: Who would like to volunteer to put the chair?
(Natalie puts her hand on...)

T: Yes, Natalie...

N: (Natalie comes to where the chairs are and she starts measuring the gap between the two chairs by counting her steps. Then by using the number of steps measured, she places the third chair, forming a translation.)

121: T: Okay, Thank you. Did you see what Natalie did? (He refers to the whole class.)
Can you show us one more time Natalie, because it is such a good idea. Make sure what Natalie does.

(Natalie shows again.)

Would somebody do it in a different way?

125: Can I take another chair? Like Lucy's chair. Thank you Lucy.
Where this one goes?
Yes, Max... Where would you put this one? What do you do to know where to put it?

M: .... (Max says something but is inaudible.)

130: T: Can you go to show me what you measure?
You measure from which point to which?

M (Max does the measurement... the class observes.)

T: From the right chair leg, to where you gonna measure it?
M: .... (Max says someting... inaudible.)

T: The same point (seems that he repeats Max).

135: T: (Then he asks the class.) Did they use the same methods?

Ps: No..... (many voices)

T: Can anybody gives me words to describe what is the difference between the methods?

P: She measures distances between the chairs... (a boy)

T: So, what is the simple word to describe what distance she measures.

140: P: From the front to the back... (a different boy)

T: So, she measures from the front of this chair to the back...

T: Is that what Max did?

Ps: No..... (many voices)

T: What distance did Max measure?

143: P: From the back to the back.
T: He measured from the back of the chair to the back, didn't he?
From same point to same point...

T: You have the two different distances in your head? Yes? Can Natalie create the translation? Can she get the shape in the right position?

150: Ps: Yes..... (many voices)

T: Can Max get the shape in the right position?

Ps: Yes..... (many voices)

T: But they are two different distances.

T: Which is the period of translation?

P: Max's..... (many voices)

T: Max's. But Natalie's method is just as valid to create a translation.
So, if we say for the period of translation we are talking about moving points to a new position.
Can anyone think of how I could move the chair? If I move all the points by a period of translation where do I place this chair?

P: (One pupil says something... but inaudible.)

T: So, when I talk about the period of translation, is like picking up this chair and place it on top of the next one. All points go one on top of the other. As Max's did.

T: There is nothing wrong with what Natalie did, because she can actually make a translation.

The above episode in using chairs wasn't part of the devised activities, but nevertheless, it is part of how the teacher implemented the devised activities. It was considered a significant event of the implementation of the activities because of its genuity and spontaneity. It also seemed to have great value for the teacher, who decided to alter his planning and use the example with the chairs. It was therefore felt important to explore the reasons underpinning this action.

Study of the Constructive Approach in Mathematics Education

Tadao Nakahara and Masataka Koyama
Hiroshima University, Japan

This paper studies an approach to mathematics lessons that enables children to actively construct mathematical knowledge. In Section 1, we give the five fundamental principles of the "Constructive Approach". Section 2 gives consideration to mathematical epistemology from a constructivist standpoint. The teaching-learning process model is studied in Section 3. Section 4 consists of a basic study of important methodological principles for teaching-learning in the constructive approach. Section 5 sets up the "lesson planning model" in the constructive approach. We perform a practical study of the constructive approach in Section 6.

Five principles of the "Constructive Approach"

This paper studies an approach to mathematics lessons that enables children to actively construct mathematical knowledge - mathematical concepts, principles, rules etc. More concretely, the aim of this thesis is to establish a theory for constructing mathematics lessons based on the following five principles that are entitled the "Constructive Approach".

CA 1. Children acquire mathematical knowledge by mental constructions of their own.

CA 2. Basically, children construct and acquire matematical knowledge in the process of being conscious, operational, mediative, reflective and making agreement.

CA 3. In the process in which children are constructing mathematical knowledge, operational activity and reflective thinking play major roles.

CA 4. Children construct, criticize and refine mathematical knowledge through constructive interaction with their teachers or with other children and then agree that it is viable knowledge.

CA 5. While children are constructing mathematical knowledge, five modes of representation, i.e. realistic representation, manipulative representation, illustrative representation, linguistic representation and symbolic representation play important roles.

CA 1 in the above is an epistemological principle based upon constructivism which was established by Piaget (1966, 1970). New developments of constructivism are shown today by von Glasersfeld (1981a, 1987, 1988), Cobb (1989, 1991), Steffe and Wood (1990), Ernest (1991) and others.

CA 2 is a principle of a teaching-learning process based upon studies made by Herscovics and Bergeron (1984, 1988), Pirie and Kieren (1989) and others who are currently studying the recognition and understanding processes of mathematical knowledge from the viewpoint of constructivism.

CA 3 is a methodological principle of teaching-learning, based upon studies of abstraction, recognition and understanding of mathematical knowledge which were performed by Piaget (1966, 1970), Skemp (1971, 1987) and others.

CA 4 is a methodological principle based upon studies of social interaction which were performed by constructivists (Cobb, 1991; Kamii, 1985; Yackel et al., 1990) and didactic scholars (e.g. Yoshimoto, 1974).

CA 5 is also a methodological principle based upon studies of representation by Bruner (1966), Lesh et al. (1983, 1987) and others.

In this way the study on the subject of this thesis "Constructive Approach" is characterized by its being based upon the study of constructivist mathematical epistemology, the constructivist study of methodological principles of teaching-learning, the constructivist study of recognition and understanding processes, and the study of representational modes in mathematical education. These four aspects are integrated in this study.

Consideration given to constructivist mathematical epistemology

This section considers mathematical epistemology from a constructivist standpoint and proposes a mathematical view of "assentive constructivism" which is the mathematical epistemological standpoint of this thesis.

First of all, the authors considered mathematical abstraction which is a central conceptual action as children construct and understand mathematical knowledge. In mathematical abstraction, extracted qualities do not exist in an external existence, but are constructed mentally by each subject of recognition. Therefore, mathematical abstraction differs definitely from empirical abstraction, i.e. the real nature of mathematical abstraction is "constructive abstraction" (Piaget, 1970; Kamii, 1985).

Next, we gave consideration to mathematical generalization that is closely associated with mathematical abstraction. It can be regarded as a conceptual action that expands an object of thought from set A to set B where $A \subset B$. The generalization which takes place under such conceptual action can be classified into the generalization of sets and the generalization of propositions.

We further went on to the consideration of relationships between mathematical generalization and mathematical abstraction. As a result we illustrated through concrete examples that in the same mathematical activities, trial abstraction and trial generalization coexist and interact with one another, i.e. the process in which mathematical knowledge is constructed consists of abstraction and generalization related in a complementary way.

Therefore, we studied a constructivist mathematical epistemology which is the basis of how to take up the above-mentioned mathematical abstraction and generalization. First of all the authors considered prior studies by Piaget (1966, 1970), von Glasersfeld (1981a, 1987, 1988), Cobb (1989, 1991), Steffe and Wood (1990), Davis et al. (1990), Ernest (1991), Tymoczko (1986) and others. Next, as a mathematical epistemology that constitutes a basis for mathematics education, we then proposed a mathematical "assentive constructivism" based upon the following four principles.

PC 1. Mathematical knowledge is constructed and acquired actively by the subject of recognition. It is not acquired by transmission or discovery. Enforcement by others becomes detrimental to constructive activity.

PC 2. Mathematical knowledge is constructed by thinking activities reflectively. It is then corrected and refined through social interaction.

PA 1. Viable mathematical knowledge is agreed in groups and becomes inter-subjective knowledge.

PA 2. Whether or not mathematical knowledge is correct is determined objectively based upon the agreement.

This standpoint is based upon an idea that subjects agree to what is constructed as mathematics if it is viable knowledge and further constructs mathematics from it, thereby repeating the process in a developmental manner. It differs from radical constructivism, because in assentive constructivism the correctness or not of mathematical knowledge is determined objectively according to the written agreement.

This mathematical epistemology is characterized by the fact that construction and agreement which are seen to be apparently contradictory to each other are positioned in a complementary manner. From such a viewpoint, formalism fixes its eyes upon the result of agreement, while radical constructivism pays attention mainly to the process of construction.

The mathematical viewpoint of assentive constructivism we have proposed above is also true of the school mathematics that children learn. As we have pointed out previously, children construct the knowledge of school mathematics mentally, from epistemological characteristics of mathematical abstraction and generalization. Considering this and the fact that children who are in the process of intellectual development are the subject, PC 1 and PC 2 concerning the above-mentioned constructivist principles must be more stressed in mathematics lessons.

In the meantime, it is also important to agree upon the result of what is constructed in the form of definitions, formulas and theorems, and that function should not be overlooked. Therefore, it is proper to regard mathematics lessons as opportunities in which children construct and agree upon mathematical knowledge.

Teaching-learning process in the constructive approach

This section studies the teaching-learning process that basically deals with how children construct and agree upon mathematical knowledge and sets up the lesson process model that makes up the framework of the constructive approach.

In the beginning, the authors examined prior studies such as the learning level theory by van Hiele (van Hiele and van Hiele, 1958; van Hiele, 1986) as a stage theory of the teaching-learning process, the six stages theory of mathematical learning by Dienes (1973) and the levels of physical representation by Wiebe (1980). We then examined the result of studies by Skemp (1976, 1987), Herscovics and Bergeron (1988) and Pirie and Kieren (1989).

Next, we established the lesson process model by making a comparative review of such prior studies and on the basis of theoretical consideration concerning the constructive process of mathematical knowledge. It consists of the following steps of teaching-learning activities.

P 1: Being Conscious

The first step consists of children's encounter with the source that generates mathematical knowledge which they try to construct. Here they become conscious of problems and give an outlook for a solution.

P 2: Being Operational

The second step consists of operational activities looking for a solution based upon the above-mentioned outlook and formation of a prototype of knowledge which is to be constructed.

The operational activities mentioned herein center on concrete operation, but include mental operation and symbolical operation as well.

P 2.5: Being Mediative

Being mediative is a flexible step which is set up according to teaching contents and children, where necessary. A variety of learning activities are set according to children's actual condition. Children make results of the second step public in a class. Questions having a new content are tackled and activities similar to the operational step are performed, though they are associated with the original question. And sometimes some portion of the essential part of corresponding knowledge is abstracted.

P 3: Being Reflective

Being reflective is a step at which mathematical abstraction is performed by looking back on activities in the operational and mediative steps and mathematical knowledge is constructed. In this step, what counts are reflective thinking, in particular equalization, idealization and realization.

P 4: Making Agreement

This step examines and refines the mathematical knowledge that was constructed in the reflective step, by discussions in a classroom. Viable knowledge is agreed in this way.

We have then decided to represent the above-mentioned teaching process as in Figure 1, by incorporating a feedback function.

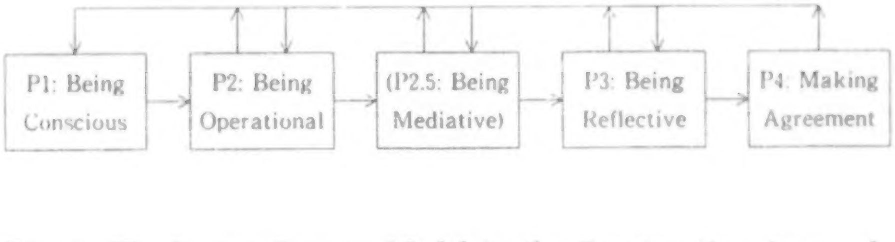


Fig. 1. The Lesson Process Model in the Constructive Approach

Methodological principles in the constructive approach

This section consists of a basic study on important methodological principles for teaching-learning in the constructive approach.

First of all, the authors studied reflective thinking and considered the views of it set up by Dewey (1933), Skemp (1971), Wittmann (1981), Piaget (1970), von Glasersfeld (1981b) and Steffe et al. (1988). Dewey and Skemp consider reflective thinking from an extensive viewpoint and do not regard such thinking as unique to mathematical activities. Here, let us call it "general reflective thinking". On the other hand, Piaget and others think that a human being performs some operational activity on concrete objects and problems, and that as a next step, they regard thinking exercised on the operational activity and the result thereof as being reflective thinking. We then decided to characterize such reflective thinking as "operational reflective thinking".

In mathematics education both types of reflective thinking are important, but what displays an intrinsic function in mathematical abstraction and generalization is operational reflective thinking.

Next, we contemplated exchanges of ideas and thoughts between children and between a teacher and children and studied how they are defined and how they function. Although they are taken in a variety of ways, such as social interaction, talk and debate (Hoyles 1985; Pirie et al. 1989), they have been defined as constructive interaction, because of interactive action constituting viable knowledge via mentally constructed knowledge.

We then gave basic consideration to the representational methods which serve as important clues and display an excellent function when children construct mathematical knowledge, as well as to the systemization of the representational methods.

We compared and reviewed prior studies by Bruner (1966), Lesh et al. (1983, 1987), Austin and Howson (1979), Haylock (1982), Clarkson (1987) and others and classified representational methods in mathematical education into five representational modes and decided to represent them as in Figure 2 (Nakahara (Ishida), 1984).

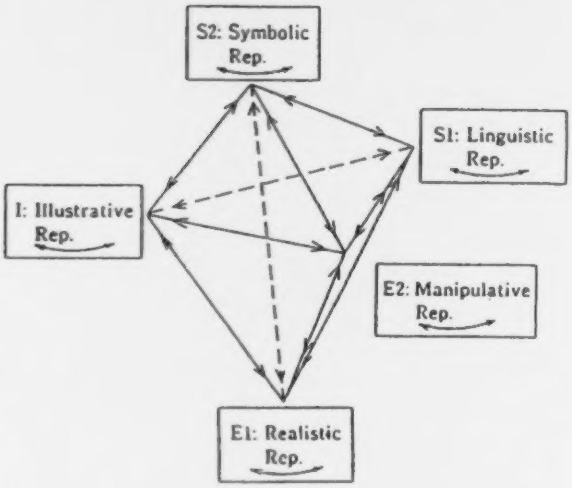


Figure 2. The (External) Representational System in Mathematics Education

This representational system utilizes the advantages of the representational system proposed by Bruner and others and covers the shortcomings when the process of constructing mathematical knowledge is clarified and where the system is applied to mathematics lessons.

We further went on to consider three representational modes unique to mathematics education, namely, manipulative, illustrative and symbolic representation, and examined them in terms of sort, characteristics and role, and identified principles of utilization in mathematics lessons for each of them.

We considered the characteristics of manipulative representation from semiotics and recognizable viewpoints. In one sense, manipulative representation can be thought to lie in the middle between realistic representation (or an alternative representation of it) and symbolic representation and to mediate between the two. We decided to call this characteristic "Intermediateness–mediateness". At the same time, manipulative representation is capable of dynamic manipulation which contributes to the solution of questions. Let us name this characteristic "dynamic manipulativeness–question solvableness".

98

Such characteristics make mathematics lessons pleasant, easy to understand, constructive and inventive, if manipulative representations are utilized. In that case, the manipulative representation contributes in particular to the construction of mathematical knowledge at two levels. One of them is to solve outstanding questions and another is to construct more general concepts, principles and rules based upon the solution of such questions. For instance, if a question is "12 – 8", the former is to give an answer, while the latter is to devise a "carry-down principle" by tracing the solution process down. The latter is of more value and in the process of it, operational reflective thinking plays an extremely significant role.

Concerning illustrative representation, the authors collected examples in textbooks and lessons, and grouped them according to their references and classified into eight types: scenic, situational, procedural, structural, conceptual, rule–relation, graphic and geometrical figure. We then brought out topologiness, freeness, similarity, visibility and metaphoricalness as basic qualities of illustrative representation.

As secondary qualities which are induced from these qualities, we pointed out intuitiveness, imagebleness, structuralness, totalism and individuality. Regarding symbolic representation, the authors identified four ways in which it is generated: linguistics, contraction, location and abridgment, When we considered results of generated symbolic representation, this led to the further characteristics of objectness, operationness and formness. We then considered the advantages which these qualities bring and listed various points such as simple–strict expression, promotion of abstraction–generalization and promotion of formal processing of mathematical information as roles unique to symbolic expressions.

Lesson planning model in the constructive approach

In this section, we summarize the results of studies made so far, and construct a lesson planning model in the constructive approach.

To begin with, the authors combined together the lesson process model with the promotion of reflective thinking and constructive interaction, and the utilization of representational modes. We then established the lesson planning model as shown in Figure 3.

Regarding the planning of lessons in the constructive approach, we construct lessons on the basis of the lesson planning model, taking account of the results of the following:

(i) studies of the contents of teaching material from the viewpoints of recognition and mathematics.

(ii) practical studies on how to make use of three methodological principles, that is CA 1, CA 2, CA 3, depending upon the contents of teaching material.

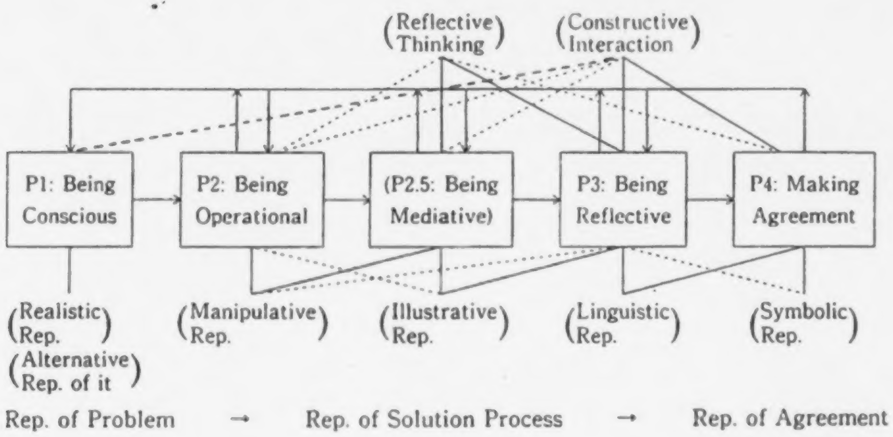

(Reflective Thinking)
(Constructive Interaction)
P1: Being Conscious
P2: Being Operational
(P2.5: Being Mediative)
P3: Being Reflective
P4: Making Agreement
(Realistic Rep.)
(Alternative Rep. of it)
(Manipulative Rep.)
(Illustrative Rep.)
(Linguistic Rep.)
(Symbolic Rep.)
Rep. of Problem → Rep. of Solution Process → Rep. of Agreement

Fig. 3. The Lesson Planning Model in the Constructive Approach

A case study of the Constructive Approach

Introduction to Fraction: Representing Fractional Parts (3rd Grade)

We made a case study of the constructive approach in teaching an introduction of fraction to the 3rd graders of primary school. It is an objective of the lesson shown below that children are aware of the possibility of representing fractional parts by an idea of division into equal parts through such an activity as placing over and folding fractional parts. Therefore, we planned the lesson taking both teaching materials of fraction and the actual state of classroom children into consideration.

Firstly, we make children use a fictitious unit named *gel* as a restriction so that they can construct various ways of representing fractional parts. Secondly, we give children two different fractional parts being based on the fact that as a result of our prior investigation the presentation of not one but two different fractional parts is effective for children to be aware of representing fractional parts by making a comparison between them. Moreover, we use 3/5 cup of juice and 2/5 cup of juice as two fractional parts and two rectangular figures of them in order to make it possible for children to notice the sum of them or the difference between them, and to place over and fold them. Thirdly, we put great emphasis on connecting children's various ways of representing fractional parts. In a whole-classroom discussion, we ask children to report their own solution for representing fractional parts and help them examine and refine their solutions by discussing about the commonness and difference among them and share the value of various ways of representing fractional parts with peers in the classroom.

Sketch of the lesson

[P1: Being Conscious]

In the first step, in order to make an emphasis on the process of abstracting fractional parts, a classroom teacher told a story using two different bottles of juice, poured each of them into four same-sized cups, and then introduced rectangular figures to represent the volume of them. The teacher planned to make children use a fictitious unit named *gel* as a measure of volume. Therefore, he began to tell a fantastic story to his children.

Teacher: Today, let's make a space travel!
Children: Yes! Lets's go!
Teacher: Four, three, two, one; fire!
(for a while)
Teacher: Now, we have arrived at a jellyfish-planet. Jellyfish-aliens welcome and give us two different bottles of juice. (He showed two bottles of juice. He poured each of them into four same-sized cups, then he introduced rectangular figures to represent them. He prepared 1 3/5 cup of blue-colored juice and 1 2/5 cup of red-colored juice. But the children were given no detail information about their fractional parts.)
Teacher: Let's inform jellyfish-aliens of the volume! You should pay attention to the fact that jellyfish-aliens use a unit

named *gel* as a measure of volume. But they understand such numerals as 0, 1, 2, 3, 4,...

Children: Only *gel* ?

Teacher: Yes! So jellyfish-aliens can't understand units of measure with which you are familiar.

Children: Does it mean we can't use units such as *dl* or *ml* ?

Teacher: Yes! You can't use those units on the earth.

C1: How should we do to inform them of the volume? If we could use such units as *dl* and *ml*, it would be easy. But it is really difficult.

C2: We had better make another convenient unit named *bal*.

Teacher: Inform them of the volume of juice somehow.

[P2: Being Operational]

In the next step, the teacher gave each child a sheet of paper on which there were two rectangular figures for two fractional parts of juice as shown in Figure 4. The children individually did operational activities in their own way using the paper for a while. Some children measured the length, while others cut two fractional parts out of paper, placed one figure over another, and folded them. During the children's activities, the teacher was observing.

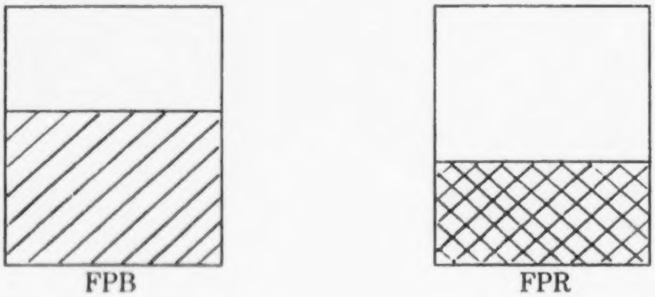


Fig. 4. Rectangular figures for fractional parts

The following shows some types of solution for representing fractional parts (FPB and FPR) that children invented as a result of their activities.

(i) FPB is half and just a little more. FPR is just a little less than half.

(ii) The length of FPB is 4 *cm* 7 *mm*, so it is 4 *gel* and 7 *gel*.
(iii) We had better make another unit named *deci-gel* as a smaller unit than *gel*.
(iv) FPB is less than one *gel*, so it is zero *gel*.
(v) FPB is zero *gel* and 47 *deci-gel*.
(vi) The sum of FPB and FPR is one *gel*.
(vii) When I divide the whole length of a rectangular figure by the unit of 1 *cm*, it can be divided into eight equal parts. FPR is equal to just three parts, so it is zero point three *gel*.
(viii) When I place FPR over FPB and fold them by noticing the difference between them, FPB is equal to three parts and FPR is equal to two parts.
(ix) When I divide the whole length of a rectangular figure into ten equal parts, FPB is equal to six parts and FPR is equal to four parts.

[P3: Being Reflective]

In the following step, the teacher asked his children to report their own solution for representing fractional parts, and organized a whole-classroom discussion in order to help children examine and refine their solutions.

Teacher: Now, we inform jellyfish-aliens of the volume of fractional parts.
C3: I can say that FPB is half and just a little more and that FPR is just a little less than half.
C4: I have a question. I think the expression such as just a little more/less is not clear.
Teacher: Indeed, so it is. But, how should we do?
C5: May I use centi-meter?
Teacher: How do you think?
Children: We can't use centi-meter.
Teacher: Yes, you are right. We can use only *gel* as a unit. Is there another idea?
C6: I have a good idea. FPB is 4 *gel* and 7 *gel*. FPR is 3 *gel* and 1 *gel*.
Children: What does it mean?
Teacher: Anyone who can explain this idea?
(for a while)
C7: Oh, I see. I'm sure that he measured the length with a ruler.

Children: Yes, we agree with you. The length of FPB is 4 *cm* 7 *mm,* so he changed both units of *cm* and *mm* to *gel.*

Teacher: I see, so it is...

C8: Why do you use the same unit *gel* ? They, *cm* and *mm,* are different units.

C9: Because we are allowed to use only *gel* as a unit!

C10: I say FPB is 4 *gel* and 7 *deci-gel*. Because the units of *cm* and *mm* are different and we already learned other units of liter and deci-liter for measure of volume.

Children: That is a good idea. It is easy to represent fractional parts with such a new unit of volume.

C11: But, I think that the expression 4 *gel* and 7 *deci-gel* is not reasonable, because 4 *gel* is more than 1 *gel.*

C12; So my idea is better than it. FPB is less than 1 *gel,* so it is zero *gel*. Because we know that the number smaller than 1 is 0.

C13: Your idea of zero *gel* is not reasonable, because it means that there is nothing.

C14: Then, zero *gel* and 47 *deci-gel* is a more reasonable expression.

Teacher: I see. Any other ideas?

C15: When I divided the whole length of a rectangular figure by the unit of 1 *cm,* it was divided into eight equal parts and FPR was equal to just three parts. So FPR is zero point three *gel.*

C16: What does it mean?

C17: It means that FPR is less than one *gel* and that FPR is equal to three parts.

C18: I think that the expression such as FPR is zero point three *gel* is a same expresson as FPB is zero *gel* and 47 *deci-gel*.

C19: Even when we divide the whole length into ten equal parts, could you still say FPR is zero point three *gel* ? If FPR was equal to four parts in case of ten-parts division, would you express it as zero point four *gel* ?

Teacher: You mean that we can say FPR is zero point four *gel* in such a case as it is equal to four parts.

C20: I did not consider such case.

Teacher: Do you understand what he wanted to do?

C21: It is similar to my idea, but I divided the whole length of a rectangular figure into five equal parts. It is a reason that when I placed FPR over FPB and folded them by noticing the difference between them, each of them could be divided into five equal parts. Then, FPB is equal to three parts and FPR is equal to two parts.

[P4: Making Agreement]

In this step, the teacher encouraged some children to report their own way of representing fractional parts by a division into equal parts, and aimed to help all children be aware of the possibility of representing fractional parts by an idea of division into equal parts.

C22: When I divided the whole length into ten parts, FPB is equal to six parts and FPR is equal to four parts.
C23: Mr., I think there are many ways of division into equal parts.
Children: Yes! There are many ways.
Teacher: Oh, there are many ways?
C24: Yes! There are as many as we like by multiplying the number of division, for example 8, 16, and 24 or 5, 10, and 15 etc.
Teacher: So, in the next lesson, we will continue to investigate the way of representing fractional parts less than one *gel* by using this idea of division into equal parts.

In this experimental lesson, the restrictions imposed and two rectangular figures with 8 *cm* length given to children could make it possible that children invent a new unit and construct various ways of representing fractional parts less than 1 *gel*. As a result of their operational activities with those figures followed by the constructive interaction in a whole-class discussion, children by themselves examined and refined their solutions, and finally they could be aware of the possibility of representing fractional parts by an idea of division into equal parts.

Final remarks

The authors consider that this study has the following meanings.

(i) This study will be able to review and to change the teacher's belief that mathematical knowledge has universality and absoluteness and is possible to be transmitted to children.

(ii) The lesson planning model improves lesson planning that is based upon discovery learning and problem solving learning, by enabling the teacher to construct lessons mainly consisting of the meaning of mathematical knowledge.

(iii) This study offers an opportunity to construct mathematics lessons that develop children's autonomy and individuality.

The remaining tasks include the experimental and positive studies of lessons over long periods and more detailed studies concerning the agreement of mathematical knowledge. The authors intend to proceed with such studies and make the "Constructive Approach" more substantial.

References

Austin, J. L. and Howson, A. G, (1979). Language and Mathematical Education. *Educational Studies in Mathematics 10,* 184-197.

Becker, J. P. (Ed.) (1989). *Mathematics Teaching in Japanese Elementary and Secondary Schools: A Report of the ICTM Japan Mathematics Delegation (1988).* Southern Illinois University.

Bruner, J. S. (1966). *Toward a Theory of Instruction.* Cambridge, MA: Belknap Press.

Clarkson, S. P. (1987). *A Study of the Relationships among Translation Skill and Problem-Solving* U.M.I. Dissertation Information Service.

Cobb, P. (1989). Experiential, Cognitive and Anthropological Perspective in Mathematics Education. *For the Learning of Mathematics 9* (2), 32-42.

Cobb, P. (1991). Reconstruction Elementary School Mathematics: 1990 John Wilson Memorial Address. *Focus on Learning Problems in Mathematics, 13* (2), 3-32.

Davis, R. B., Maher, C. A. and Noddings, N. (Eds.) (1990). *Constructivist Views on the Teaching and Learning of Mathematics.* Journal for Research in Mathematics Education Monograph No. 4, Reston, VA: NCTM.

Dewey, J. (1933). *How We Think: a restatement of the relation of reflective thinking to the educative process.* Lexington, MA: D. C. Heath.

Dienes, Z. P. (1973). *The Six Stages in the Process of Learning Mathematics*, translated from the French by P.L. Seaborne. Windors, Berks: NFER.

Ernest, P. (1991). *The Philosophy of Mathematics Education*. London: Falmer.

Haylock, D. W. (1982). Understanding in Mathematics: Making Connections. *Mathematics Teaching 98*, 54-56.

Herscovics, N. and Bergeron, J. C. (1984). A Constructivist vs. a Formalist Approach in the Teaching of Mathematics. *Proceedings of the 8th PME*, 134-142.

Herscovics, N. and Bergeron, J. C. (1988), An Extended Model of Understanding, *Proceedings of the 10th PME-NA*, 15-22.

Hoyles, C. (1985), What is the Point of Group Discussion in Mathematics? *Educational Studies in Mathematics 16*, 205-214.

Kamii, C. K. (1985). *Young Children Reinvent Arithmetic: Implications of Piaget's Theory*. New York: Teachers College Press.

Lesh, R. and Landau, M. (Eds.) (1983). *Acquisition of Mathematics Concepts and Processes*. New York: Academic Press.

Lesh, R., Post, T. R. and Behr, M. J. (1987). Representation and Translations among Representations in Mathematics Learning and Problem Solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, NJ: LEA.

Mayberry, J. (1983). The van Hiele Levels of Geometric Thought in Undergraduate Preservice Teachers, *Journal for Research in Mathematics Education 14*, 58-69.

Nakahara (Ishida), T. (1984). A Study on the Representational Systems in Mathematics Teaching, *Proceedings of the 8th PME*, 238-243.

Nakahara (Ishaida), T. (1992). *Study of the Constructive Approach in Mathematical Education*, A Dissertation recognized by Hiroshima University.

Piaget, J. (1966). *Mathematical Epistemology and Psychology*: Part Two. In E.W. Beth and J. Piaget, *Mathematical Epistemology and Psychology*, translated from the French by W. Mays. Dordrecht: Reidel.

Piaget, J. (1970). *Genetic Epistemology*, translated from the French by E. Duckworth. New York: Columbia U.P.

Pirie, S. E. B. and Kieren, T. (1989). A Recursive Theory of Mathematical Understanding, *For the Learning of Mathematics 12*, 23-34.

Skemp, R. R. (1971). *The Psychology of Learning of Mathematics*. London: Penguin Books.

Skemp, R. R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching 77*, 20-26.

Skemp, R. R, (1987). *The Psychology of Learning Mathematics: Expanded American Edition*. Hillsdale, NJ: LEA.

Steffe, L. P., Cobb, P. and von Glasersfeld, E. (1988). *Construction of Arithmetical Meanings and Strategies*. New York: Springer-Verlag.

Steffe, L. P. and Wood, T. (Eds.) (1990). *Transforming Children's Mathematics Education: International Perspectives*. Hillsdale, NJ: LEA.

Tymoczko, T. (Ed.) (1986). *New Directions in the Philosophy of Mathematics*. Boston: Birkhauser.

van Hiele, P. M. and van Hiele, G. D. (1958). A Method of Initiation into Geometry at Secondary Schools. In D.H. Freudenthal (Ed.), *Report on Methods of Initiation into Geometry*. Groningen: J. B. Wolters.

van Hiele, P. M. (1986). *Structure and Insight: A Theory of Mathematics Education*. New York: Academic Press.

Vinner, S. and Hershkowitz, R. (1980). Concept Images and Common Cognitive Paths in the Development of Some Simple Geometrical Concepts. *Proceedings of the 4th PME*, 181-186.

von Glasersfeld, E. (1981a). The Concepts of Adaptation and Viability in a Radical Constructivist Theory of Knowledge. In Sigel, I. E. et al. (Eds.), *New Direction in Piagetian Theory and Practice*. Hillsdale, NJ: LEA.

von Glasersfeld, E. (1981b). An Attentional Model for the Conceptual Construction of Units and Number. *Journal for Research in Mathematics Education 12*, 83-94.

von Glasersfeld, E. (1987). Learning as a Constructive Activity. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, NJ: LEA.

von Glasersfeld, E. (1988). *The Construction of Knowledge: Contribution to Conceptual Semantics*. California: Intersystems Publications.

Wiebe, J. H. (1980). Levels of Abstraction in Physical Models for Teaching Arithmetic. A 4th ICME Handout.

Wittmann, E. (1981). The Complementary Roles of Intuitive and Reflective Thinking in Mathematics. *Educational Studies in Mathematics 12*, 389-397.

Yackel, E. et al. (1990). The Importance of Social Interaction in Children's Construction of Mathematical Knowledge. In T. J.

Cooney (Ed.), *Teaching and Learning Mathematics in the 1990s*. Reston, VA: NCTM.

Yoshimoto, H. (1974). *Theory of Disciplinary Instruction* (in Japanese). Tokyo: Meijitoshoshuppan.

Basic Imagery and Understandings for Mathematical Concepts

Peter Bender
Universität-GH Paderborn, Germany

An overwhelming part of everyday mathematics teaching all over the world can be characterized by the absence of meaning for the students. The didactical conception of basic imagery and understandings (BIU) is an attempt to make mathematics teaching meaningful. This conception takes into account the ambiguity of mathematical concepts in the realms of subject matter, of epistemology and of psychology and the crucial problem of matching the concepts from different realms referring to the 'same' object. The underlying epistemology is that of social constructivism, however ascribing a predominant role to the teacher and her or his utterances.

The contribution of epistemological constructivism to mathematics education

Modern mathematics didactics is confronted with at least two profound epistemological problems which are logically independent from, but which are practically closely connected with, each other: First, there is the basic, but by no means trivial, relationship between those concepts and contents belonging to the more or less pure sphere of the discipline of mathematics, those concepts and contents which we educators want the students to acquire, and those concepts and contents which the students really cognitively construct (or, better: which we suppose the students to construct). Second, there is the fundamental question of how learners acquire knowledge and form concepts, which can be embedded in the more general question of how individuals interact with the (cognitive) world around them. By asking this question in this way I aim at the epistemological philosophy of constructivism, after which "the function of cognition is adaptive and serves the organisation of the experiential world, not the discovery of ontological reality" (von Glasersfeld 1989, 182).

Although my focus is on the first of these two problems, namely the ontological relationship between the realms of subject matters, epistemology and psychology, there is, throughout the paper, a continuous noise in the background, caused by constructivist ideas. – The extreme variant of constructivism denies the existence of commonly shared knowledge and concepts at all, and there arise fundamental difficulties, e.g. how to explain the broad, surely low, but nevertheless existing, success of school instruction, which shows in obviously largely commonly shared knowledge and concepts (despite billions of well known and sometimes well documented failures). In order to overcome those difficulties, there have been developed several modifications of radical constructivism (in the sense of its protagonist, von Glasersfeld). One of the most interesting variants seems to be *social constructivism* (in the sense of Ernest, 1994a), which postulates the pre-existence of social reality to every individual, the generative character of human conversation, and the possibility of collective mental functioning.

It is doubtful whether social constructivism actually is constructivism, as in terms of radical constructivism the social reality is part of the individual's construction of the world, and conversations, as well as seemingly collective mental processes have no objective character at all, but belong to each individual's experiential world which she or he organizes in a subjective way.

My ideas of how to bridge the gap between subject matters, epistemology and psychology are closely connected with the so called social constructivism, as can be seen in the following chapters. – But surprisingly, from a radical constructivist point of view there have to be drawn similar instructional consequences (which within this philosophy would not be called 'instructional'): It is the teacher's profession to offer the students possibilities for constructing their knowledge and concepts. Although, on principle, the teacher cannot control or fully understand the students' constructions, she or he can judge to some extent – by observing how the students talk about their knowledge and their concepts, how they apply them or how they build theories with their help –, whether the students' constructions match her or his own knowledge and concepts (which she or he has proved to be adequate in some way by mental reflections, in conversations with other people, including students, applications and formations of theories – permanently due to more or less slight changes). In this sense, good instruction is nothing else but the *preparation of* (relatively) *relevant settings* as realistic, as well as possibly artificial, parts of the students' experiential worlds, in order to stimulate their construction of knowledge and concepts.

What is more: If there actually is no direct transfer of knowledge and concepts, all the more teachers have to develop 'instructional' settings with particular care, taking into account not only subject matters, but also their students' personalities, as well as social, pedagogical etc. variables. Even if, on principle, teachers will never really know about their 'success', they are obliged to strive for arranging optimal (in whatever sense) settings. – All the implications for teaching practice which are readily drawn from constructivist perspectives, like

- sensitivity towards the learner's previous constructions,
- diagnostic teaching with cognitive conflict techniques for remedying learner errors and misconceptions,
- attention to self-regulation by learners,
- use of multiple representations,
- concern with learner cognitions as well as with teacher cognitions,
- problematizing knowledge as a whole,
- taking into account that there is no 'royal road' to truth,
- awareness of the dichotomy between learners and teacher goals,
- awareness of the linguistic basis and the social construction of mathematical knowledge,

etc. (cf. Ernest 1994b, 338f) are well coherent with my plea for careful preparation of instructional settings. Compared with this, in my opinion, the "progressive teaching ideology [which] sanctions anything the child does as expressions of its individual creativity, and naively assumes that the child can discover much of conventional school knowledge on its own" (Ernest 1994b, 337) is, if based on the philosophy of constructivism, a misunderstanding.

I will now turn to my proper subject, i.e. "basic imagery and understandings for mathematical concepts" which constitutes a substantiation of (moderate) constructivist ideas, structured along the key words 'meaning', 'basic', 'imagery' and 'understandings', explained along concrete mathematical examples, and now and then enriched with the discussion of constructivist arguments.

The problem of judging the 'adequacy' of students' 'concepts'

Despite immense progress in the field of mathematics didactics there are still a lot of mathematics educators as well as teachers who adhere to a rather narrow picture of their subject, namely consisting on the whole of abstract relations between abstract objects and some calculation. For them, intuitive, vivid, enactive or application oriented ways of doing mathematics do not belong to true mathematics, but are mere approaches. The advantage of this picture is that the contents can be identified exactly, and can easily be made accessible to presentations in textbooks, as well as to empirical research on how students handle them, or to (so called) intelligent tutorial systems.

Yet, this picture is not a suitable foundation for teaching and learning mathematics (neither for doing or applying mathematics), as in it the category of *meaning* is ignored and hence the constitution of meaning is not a matter of education. – But in every thinking or learning process the individual assigns some meaning to some notion, situation or circumstances, and teachers, in particular mathematics teachers, have to take into account these processes of assignment.

Closely connected to the difficulties in recognizing and controlling the students' learning processes is the problem of matching concepts in the realms of mathematics, epistemology and psychology (which I will call 'mathematical', 'epistemological' and 'psychological concepts' respectively). – The conception of basic imagery and understandings (BIU) offers a didactical frame for this matching problem. In German mathematics education this conception has a long tradition. Rudolf vom Hofe (1995) investigated its history and found a lot of variants in the last 200 years, most of them tackling the matching problem by designing ideal normative mathematical concepts in the epistemological mode (vom Hofe names them "basic ideas") serving as models for the students' formation of concepts in the cognitive mode (which he names "individual images").

It seemed to be natural to all those educators to found their conceptions on an analysis of subject matter and to include their rich teaching experience as an empirical background. Thus they were much closer to their students than many mathematics professors at the universities or teachers at the Gymnasiums (in former times with the top 10 % of each age-group) who taught (and often still do teach) mathematics – maybe in an elementarized, but still – in a rarely modified manner as a pure discipline. On the other hand those educators, too, did often not care for what really happens in the students' brains, and, furthermore, in spite of their good ideas their efforts had only success up to a modest level.

But one must admit that only since the 1970s there has been reasonable technology for thoroughly studying classroom actions, namely video recordings. Of course, even with this technology one does still not know how cognition 'really' works. Neither the mathematical formalization of thinking processes, nor the definition of man as an information processing being similar to a computer (Simon 1969) brought about much new insight in *human* cognition. But based on the talents of video technology we learned a lot about communicative and social interaction in the classroom, in particular, how mathematical meaning is implicitly and explicitly negotiated between the participants (cf. Bishop 1985).

Due to the constructivist and connectionist roots of their theories, some cognitive scientists underestimate, ignore or deny a dominant influence of the teacher and, consequently, of the subject matter on the students' learning processes. – In fact, during painstaking examinations of video taped and transcribed micro situations middle and long term effects can easily get out of sight. If one concentrates on social and communicational characteristics of a situation, the subject matter tends to play only a minor role. And comparing students' deviating verbal and non-verbal manifestations with teachers' obvious original intentions may support severe doubts in the efficacy (or even possibility) of extraneously determined learning processes. – These tendencies are supported by the researchers' aim to overcome the old theories because of their meager success.

On the other hand careful re-analyses of classroom situations under *subject matter aspects* often lead to plausible recasts or improvements (as well as to verifications) of former interpretations based on interaction-theoretical grounds. So to me it sounds unreasonable to exclude these aspects when exploring such a situation. As I pointed out before, in German mathematics didactics, for a lot of mathematical concepts there are well known elaborated teaching routines. Whether a teacher relies on such a routine or not: From the words, diagrams etc. that she or he uses, from her or his rejection or acceptance of students' answers etc., the observer frequently can disclose the teacher's own imagery and understandings about the mathematical concept in question. (Throughout this paper, the notion of mathematical concept includes theorems, mathematical structures, procedures etc.) – Surprisingly often, the teacher's own imagery and understandings seem to be inadequate, or at least the teacher evokes inadequate imagery and understandings among the students. – These problems can be tackled didactically with the help of the conception of BIU, which is meant to be a theoretical and practical frame for a normative, descriptive and constructive treatment of concept formation processes.

A radical constructivist would argue that there is no adequacy or inadequacy of imagery and understandings. Here we have reached a point of discourse where there might be no agreement. For me, *adequacy* of concepts or adequacy of imagery and understandings is a useful and important didactical category. Of course, adequacy cannot be proved like a mathematical theorem. Whether a student's concept is adequate even cannot be stated uniquely, neither in the prescriptive, nor in the descriptive mode. But there are strong hints in either mode: If a student's statements about, and actions with a mathematical concept sound plausible and seem to be successful to her or his own common sense as well as to experts, we would concede some adequacy (for more details cf. E.J. Davis 1978).

As stated in the beginning – from a didactical point of view, it is not crucial whether teachers actually 'teach' their students or whether they only *stimulate* their students' concept formation processes. Good 'teaching' always contains stimulating the students' own acitivities.

Explanation of the key word 'basic'

By the adjective 'basic' there are expressed several essential characteristics of the conception of BIU:

- It includes a tendency of *epistemological homogeneousness and obligation* how mathematical concepts should be understood.
- *Psychologically* speaking, it indicates that students' individual concepts normally are, and in the teaching processes the epistemological concepts should be, *anchored in* the students' *worlds of experience.*

- With respect to *subject matter* it stresses the importance of *fundamental ideas* (in the sense of Halmos' elements, 1981, or Schreiber's universal ideas, 1983) guiding the study of any mathematical discipline.

Epistemological homogeneousness: This tendency seems to be in contradiction with modern pedagogical and didactical paradigms like "the students should create their own mathematics", or "the students have to find their individual ways in solving mathematical problems" etc. In fact, teaching does not mean telling (Campbell and Dawson, 1995), but it means stimulating students' cognitive activities, negotiating mathematical meaning in the classroom etc.

But this way of conceiving the teaching-learning process does not entail any obligation for the teacher to tolerate or even to support inadequate individual concepts, on the contrary: it makes the teacher's task much more difficult. She or he must be provided with a good theoretical and practical competency in mathematics, mathematics applications, epistemology, pedagogy, psychology, social sciences etc., in order to

- develop her or his own view of the epistemological kernel (which must not be identified with a mathematical definition) of some mathematical concept which the students shall acquire,
- perceive the students' actual individual concepts as truly as possible and judge their adequacy,
- help the students, if necessary, to improve or to correct their individual concepts into adequate ones near the epistemological kernel,
- possibly learn by the students and improve her or his own individual concepts.

This task imparts a predominant role in the teaching-learning process to the teacher's own imagery and understandings and to their transposition into didactical action. For example, if for the calculation of π a circle is approximated by a sequence of polygons and the teacher uses a phrase like "in this sequence the polygons have more and more vertices, and finally they turn into the circle", the students' formation of an adequate concept of limit is obstructed.

The epistemological kernel of a concept corresponds to a commonly shared socio-psychological kernel. Such a socially constituted kernel is an important prerequisite for the construction of individual argumentation and its introduction again into classroom interaction (cf. Krummheuer, 1989). – It is obvious that this commonly shared kernel should be as extensive as possible, which, again, gives the teacher a central position in the teaching-learning process.

Anchoring mathematical concepts in the students' worlds of experience: Even working mathematicians need some real world frame for doing mathematics ("we consider ...", "if x runs through the real line ..." etc.; cf. Kaput, 1979, and many others). All the more students need such frames so that they can constitute meaning with the subject matter they are about to learn (Davis and McKnight, 1980; Johnson, 1987; Fischbein, 1987, 1989; Dörfler 1996 etc.). As such frames do not belong to the epistemological concepts, the teacher is rather free when constructing real world situations where basic imagery and understandings can be unfolded.

These situations need not be absolutely realistic, on the contrary, by alienating them with the help of fairy-tale traits and concentrating on the essence they can be turned into metaphors with their explanatory power. One can take human beings, animals, things, which are more or less anthropomorphized and more or less mathematized. These participants of the situation have to act somehow, following some arbitrary rules, pursuing some arbitrary plans, obeying arbitrarily physical and other natural laws, or not.

For a lesson about the integral as area function for a given function I designed the following situation: The x-axis is a hard-surface road; north of this road (in the coordinate system) there is a uniformly wet swamp which is bounded by the road and the graph of the function in question. A vehicle drives on the road in positive (eastern) direction, with an arm perpendicular to the road which is sufficiently long to reach all parts of the swamp during the trip. With the help of this arm the water is *absorbed* uniformly from the swamp (on the basis of some uninteresting technology) and collected in a cylindrical jar. Thus, at any moment the level of the water in the jar is a linear measure of that part of the area which has already been passed by the vehicle.

If the vehicle reaches a position where the function is negative, the metaphor has to be extended: South of the road there is a desert which is bounded by the road and the negative parts of the graph and which has to be *watered* uniformly by the vehicle. For this purpose the vehicle has a second arm perpendicular to the road which is sufficiently long to reach all parts of the desert during the trip. Again, the exact mechanism is not interesting; the only important thing is that the level of the water in the jar drops proportionally with the desert area passed.

Of course, this metaphor contains a lot of technical and didactical problems which have to be considered thoroughly: – What happens if the jar is full (empty) and there is still swamp (desert) area to be drained (watered)? – Draining the swamp and watering the desert have to be accomplished with the same velocity of flow (whatever this physical notion means). – On principle, one needs a new coordinate system for the function of the water level (the integral function). – When the vehicle makes a half turn and then drives in the negative (western) direction, the two arms change their positions, and now the desert has to be in the north and the swamp has to be in the south of the road (in accordance with the mathematical changing of positive and negative area). – But if the starting point of the vehicle is finally made a variable, the efficiency of the metaphor comes to an end.

Every metaphor has its limitations (cf. Presmeg, 1994), but this is no drawback. The one which I just described should make plausible

- continuous measurement,
- the transfer from area measurement into linear measurement and
- the concept of negative area.

It thus appeals to common sense, and if the teacher wants the students to maintain their common sense, it is a must to emphasize the limitations of any metaphor.

Situations which are appropriate for mathematics teaching rarely come along by themselves. Genuine mathematics applications are often not suited for supporting concept formation, as they are frequently overloaded with alien problems. At the same time the teacher should not evoke the impression that some artificial situation, designed for the use in mathematics teaching, would be an example for genuine mathematics applications. Sometimes, this coincidence can happen, but usually it does not; and students with common sense realize the artificial character of such a situation.

Fundamental ideas for mathematical disciplines (in an epistemological and psychological sense): Basic imagery and understandings are not only meant as a peg on which to hang some mathematical content, but they shall lay the foundations for further meaningful interpretations of concepts within a mathematical discipline.

Imagery and understandings

The notions of imagery and understanding stand for two fundamental psychological constructs. There exists an extensive literature about them. Different authors have different definitions, most of them not very concise. A lot of contemporary cognitive scientists disregard these two constructs anyway, as they escape hard empirical research and do not fit a computer related view of intelligence. – But it is just these shortcomings (seen behaviouristically), their vagueness and flexibility, which turn these constructs into suitable means for analyzing (and promoting) such complex-didactical objects as human teaching-learning processes.

Imagery can be grasped as: mental, often visual (but also auditory, olfactory, tactile, gustatory and kinesthetic; cf. Sheehan, 1972) representations of some object, situation, action etc. having their sensory foundations in the long term memory and being activated in *conscious* processes. A person activating some imagery has already some meaning, some intentions in mind and organizes these processes according to these intentions (Bosshardt, 1981). – Imagery is closely related to intuitions, but its objects are more concrete, and meaning plays a more important role.

The objects of imagery (and understandings) can be given in different modes, namely analogous or propositional. I don't want to resume the cognitive scientists' quarrel in the 1970s about the interrelations between these two modes or about their separate existence as ways of thinking. In my opinion both are valuable means for analyzing imagery and understandings in teaching-learning processes.

Apparently, imagery is more closely connected to the analogous mode, and understandings are more closely connected to the propositional mode of thinking. But it is difficult for a person to activate some imagery without propositional elements, in particular in didactical situations, as in these situations verbalization is *the* fundamental means for a participant to communicate either with others or with her- or himself (this communication with oneself being a transposition of a social situation to one's mind which is typical for teaching-learning processes). On the other hand, there can be no process of understanding without recurring to any plausible imagery and to analogous elements.

Obviously, thinking in the analogous mode can be stimulated by analogous means like pictures, diagrams etc. (with a lot of limitations; cf. Presmeg, 1994), and the propositional mode can rather be stimulated by propositional means like verbal communication. In the age of paper and pencil and of books, analogously given objects frequently are of a visual, static nature, and the learners have to undertake some effort to make these 'objects' plausible, meaningful, vivid imagery matching their worlds of experience. In the nearest future, the use of multimedia in schools (in the western world) possibly will relieve the students from these efforts.

Whether multimedia will be conducive to the students' learning processes, is not yet settled: The students' inclinations and abilities to undertake efforts to generate mathematical concepts could be undermined. – This problem is complementary to the following classical one, related to the use of visualizations (diagrams, icons etc.): Among educators there is a naive belief that visualizations do facilitate the students' learning processes. But as, for example, Schipper (1982) showed with primary graders, many visualizations are not self-explanatory at all, but they are subject matter which has to be acquired for its own sake on the one hand, and in relation to the visualized contents on the other hand. – As a matter of course, visualizations can be successful didactical means, but not because they would reduce necessary effort, but because they demand more effort and give hints how to direct and structure this surplus effort and thus make it effective.

There are didactical situations, as well as mathematical concepts, as well as students, for which resp. for whom one of both modes is more suitable. For teaching and learning mathematics it is important that there has to be a permanent transformation between the two modes. Maybe geometry can be treated predominantly in the analogous mode, and algebra in the propositional mode, maybe the teacher is even able to take into consideration the preferences of single students. But on principle, both modes must be present.

Taking into account the wide-spread propositional appearance of mathematics teaching, in particular on the secondary level, there is need of an increased use of the analogous mode all over the world. – By stressing the students' anchoring of their individual concepts in their worlds of experience, the conception of BIU lays some accent on the analogous mode, as a prophylactic counterweight to the preponderance of the propositional mode in the upper mathematical curriculum.

The psychological construct of *understanding* is still more complicated, non-uniform and, from a constructivist point of view, questionable. For didactical reasons the following aspects are relevant:

(1) One can understand people, their actions, situations, the motives or the aims of the participants (practical knowledge of human nature, *common sense*).

(2) One can understand utterances *medially* and *formally* (e.g., if they are made loud enough and in a language one knows).

(3) One can understand the *content* of a *message* made by someone (understand what this someone *means* by a certain communication, text, phrase, word, symbol, drawing etc.).

(4) One can understand technical matters, working principles of gadgets, mathematical structures, procedures etc. (*expertise*).

At first glance, aspect (4) seems to be most suitable for the conception of BIU. But it becomes immediately clear that each of these aspects is important for the learning of mathematics and has to play an essential role in the conception, in particular (3). This aspect is a classical psychological paradigm, but the general opinion about it has changed, not least under the influence of constructivism: Today, one does not believe anymore that it consists just of finding some objective meaning of given signs, but that the receiver of a message tends to and has to embed the message in some context and, in doing so, tries to reconstruct its meaning (cf. Engelkamp, 1984), thus getting near aspect (1).

It goes without saying that there is no understanding (3) without (2): the sender and the receiver of a message have to have a common language, not only in a direct, but also in a figurative sense: As Clark and Carlson (1981) put it, there has to be a "common ground", which, again, refers to aspect (1). – In school teaching, and in particular in mathematics teaching, the common ground of teachers and students is often rather thin, if existing at all. – But, extending the common ground does not only mean that the students have to be better instructed so that they make the teachers ground their own. Rather the teacher must engage in the students, attach importance to them (and not only to the subject matter), understand them as human beings (again, aspect (1)), and try to reconstruct or to anticipate their ways of thinking.

By following the conception of BIU, to some extent the teacher is *forced* to do so, and furthermore, her or his expertise can be promoted. But this way of teaching and learning demands much more effort for both parts, in comparison with the ususal way, where teachers, in good harmony with the students, are satisfied with students' instrumental understanding (in the sense of the late Robert Skemp, 1976).

In the following example, the teacher (resp. the researcher) did not quite understand the student's ways of thinking. It was originally described by Malle (1988) and re-analyzed by vom Hofe (1996):

In order to develop the concept of negative numbers, Ingo, the student, was given the following situation: "In the evening the temperature is 5 degrees (Celsius) below zero. During the night a warm wind moves inland, and the temperature rises *by* 12 degrees. – What is the temperature next morning?" Ingo answers correctly: "7 degrees", but in the dialogue with the interviewer, he shows inadequate imagery. When he shall sketch a picture of the situation, he asks whether he must draw three thermometers, and later he explains that at midnight the temperature went up to +12 degrees, and in the morning it dropped to +7 degrees.

Malle gives well known and, of course, correct explanations for Ingo's obvious inadequate dealing with the situation: Ingo is not able to identify the elements which are important for solving the problem, but invents additional information and tells fairy tales, and he does not differ between the starting and the final state (i.e. the starting and final temperature, represented on the thermometer) on the one hand and the change between the states (the rise of the temperature) on the other hand.

In his careful re-analysis, vom Hofe shows that the problem lies in Ingo's imagery about the physical situation, which is no suitable basis for the formation of the mathematical concept. Whereas the interviewer expects Ingo to focus on the changes of the mercury column (as a direct model of the number line), Ingo imagines two masses of air, a cold and a warm one, which mix and result in a third mass with average temperature. Therefore he needs *three* thermometers, and in the night the temperature does not rise *by* 12 degrees, but *up to* 12 degrees, and goes down again in the morning. The idea of mixing air masses is, physically speaking, not at all inept, but it merely does not fit the mathematics that the interviewer has in his mind. For Ingo, there are two states of temperature which result in a third one, the weighed arithmetical mean, and not one state which changes into another.

Granted that every human being tends continually to conceive, or to make and to keep her or his environment meaningful and sensible, one must admit that usual mathematics teaching in large parts has a contra-productive effect. – The strive for "constance of meaning" (Hörmann, 1976) is in my opinion a characteristic trait of humans, which, for example, is largely ignored in Piaget's biologistic theory of equilibration.

Mathematics teaching, too, is such an environment which humans who are in touch with it try to make meaningful and sensible. As an *extreme* example, (in a famous French movie from 1984) in a physics lesson in Paris the absent-minded student from Algeria understands "le thé au harème d'Archimède", when he hears "le théorème d'Archimède" (which means: "tea time in Archimedes's harem" instead of "the theorem of Archimedes"). Even if we omit such extreme cases, it still seems to be rather normal all over the world that students tend to develop their own non-conformistic imagery and understandings, which, however, often remain implicit.

Fischbein (1989) calls them "tacit models" and characterizes them as simple, concrete, practical, behavioural, robust, autonomous and narrowing. Their robustness results from their simplicity, their anchoring in the students' worlds of experience, and their short term success with convenient applications (see the example of Ingo and the temperature). Inadequate tacit models come into being because of lack of adequate basic imagery and understandings, which in their turn would also be concrete, practical, etc., successful and therefore robust, and not narrowing, but capable of expansion. So the conception of BIU includes the strategy of occupying the students' frames with adequate basic imagery and understandings from the beginning, i.e. to give them the possibility and to enable them to develop such imagery and understandings by themselves.

Nevertheless, students will still generate a lot of inadequate tacit models, and teachers must be able to recognize them and to help the students to settle them. In this, again, the teachers can be supported by the theoretical and practical frame of the conception of BIU, thus using the constructive aspect of the conception (as vom Hofe, 1995, puts it).

Fischbein (1989), like many other educators and cognitive scientists, recommends that the students should undertake meta-cognitive analyses in order to discover and eliminate the defects in their frames. – I couldn't find evidence in the literature that students would be able to successfully analyze their own (wrong) thinking without massive interventions by the teacher or by some interviewer. According to my own experience with young people in all grades, they are overstrained if they shall reflect reflexively about their own reflections.

Indeed, in many classroom situations there can be found actions of understanding on a meta-level; for example, if students recall how they solved a certain problem, or if they try to find out the teacher's intentions, instead of trying to understand the contents of her or his statements. But, in general, this kind of understanding (aspect (1)) is not explicitly reflected by the students.

One essential trait of every didactical situation is (or should be) that the participants strive for understanding the contents of some message given, verbally or non-verbally, by the teacher, students, the textbook etc. (aspect (3)), with the underlying aim that the students shall acquire expertise (aspect (4)). Whereas aspect (3) stresses the *processes* of understanding, aspect (4) stands for the *products* of these processes. The products are not only results, but at the same time they are starting points for new processes, and each understanding process starts on the ground of some already existing understanding.

In mathematics teaching, both aspects of understanding ((3) and (4)) deal with the same objects: the messages, seen ideally, deal with mathematical concepts, about which the students shall acquire expertise. – In the humanities and in the social sciences, as well as (in an indirect way) in mathematics, this expertise again often refers to social situations (in a wide sense) and thus is in parts identical with aspect (1). – So, finally, in normal teaching-learning processes all the aspects of understanding discussed here belong together and are essential for success.

In my view, there is no understanding without imagery, and no imagery without understanding. With the notion of imagery there are stressed the analogous mode, roots in everyday lives, intuitions etc., – whereas with the notion of understanding there is laid some accent on the propositional mode, on subject matter, on predicates etc., – but both notions do not only appear together, rather they have a large domain of essence in common.

Examples from mathematics

There can be identified roughly four types of BIU for the use in mathematics teaching in the primary and secondary grades:

A. More or less *global* BIU, especially for the formation of the concept of number and for elementary arithmetic: multiplication as repeated addition; division as partitioning (splitting up; 'Aufteilen') or distributing (sharing out; 'Verteilen'); fractions as quantities or as operators, negative numbers as states or as operators, the machine model for operators, the little-people metaphor for running through an algorithm. Basic imagery and understandings are not bound to primitive, non-quantifiable actions (in the sense of intuitive understanding according to Herscovics and Bergeron, 1983), and their formation is not a kind of mathematical propedeutics or pre-mathematics, but – in my opinion – genuine mathematics (just without calculus with symbols). They would be useful in the upper secondary grades as well, for example with the concept of limit and infinitesimal thinking as a whole.

B. More or less *local* BIU, e.g. the arithmetical mean, the internal rate of return of an investment, the circumcircle of a regular polygon.

C. BIU for *extra-mathematical* concepts, situations, procedures (from physics, economics, everyday lives etc.), which are to be used in mathematics teaching (example: Ingo and the temperature).

D. BIU for *conventions*, e.g. the meaning of symbols, or of diagrams. Example: The teacher tries to explain subtraction with the help of the following situation: "Mother baked six cakes for her daughter's birthday; the dog Schnucki ate four of them. How many are there left?" She draws 6 circles on the blackboard and crosses out four of them (each with one line), hoping to support visually the understanding of the problem 6–4=2: But Ralph, a learning disabled child, wonders why the teacher halves the marbles (Mann 1991).

It goes without saying that the prototypes, metaphors, metonymies (Presmeg 1994) used for BIU should not obscure the concepts they refer to, like in the following example:

Euclidean geometry: As a preparation for proving the existence and uniqueness of the incircle of a triangle, the teacher asks the students: "Imagine a cone with several balls of ice cream intersecting each other physically, and a plane section through the cone containing its axis of rotation. Do so three times, by identifying the tip of the cone with the three vertices of the triangle one after another." – A more suitable imagery would be: to stick a small circle near one vertex between its adjacent edges. When the circle is blown up like a two-dimensional balloon, it moves away from that vertex, still touching the two edges, until it meets the third edge and thus reaches its final position as incircle. In dissociation from the pure Euclidean way of doing geometry, this metaphor makes use of kinematic and continuous physical phenomena from the students' worlds of experience.

Of course, mathematical concepts should not be falsified, as it is the case with the concept of circle in Papert's (1980) original idea of *turtle geometry*. The children shall draw a circle by programming the turtle to do a straight motion of length 1, then to do a right turn of the amount 1, and to repeat these two actions 360 times, i.e. by drawing a fuzzy regular polygon with 360 vertices. In fact, the result looks like a circle, but the way in which it was produced belongs to a concept which is essentially different from the Euclidean circle. It's true, that every line on the computer screen is a sequence of squares; but this is not the point, as students with some experience with paper and pencil as well as with computer screens will recognize the shortcomings of any realization of geometric forms and will be able to idealize these forms, if at least the underlying activities are appropriate. – But the procedure for making a Logo circle is not appropriate for Euclidean geometry. – Furthermore, I doubt that the Logo geometry is a good preparation for differential geometry – eventually it is a helpful model for someone who already has the concept of mathematical limit at her or his disposal, whereas it is likely to be a mental obstacle for someone who is still on her or his way of acquiring this concept, let alone for primary graders.

Transformation geometry: When in the late 1960s and early 1970s transformation geometry was pushed into the mathematics curriculum, it was assumed that real motions of real objects could serve as BIU. In fact, the students accepted these BIU willingly and transferred them easily into continuous motions of point sets in the plane. But the crucial point was the abstraction of the motions, which the students in general did not manage to perform. Their BIU of transformations grounded on motions were so robust that the good advice to focus their attention to the starting and final positions of the geometric forms or to the plane as a whole remained useless, because, for example, the notions of starting and final position, again, evoked imagery about motions (cf. Bender, 1982).

Thus, mathematicians and mathematics educators failed to establish in the curriculum the full algebraization of geometry by transformation groups, and up to today geometric transformations are not treated as objects on their own, but only as means to investigate geometric forms. The idea of embodying Piaget's groupings of thinking schemes in geometric transformation groups had proved to be too naive.

By the way, there are reasonable didactical applications of continuous motions, e.g. in good old *congruence geometry* by Euclid and Hilbert: Two geometric forms are conguent, if they can be moved to each other in a way that both exactly cover one another. In German there is a synonym for the word 'kongruent' which is due to this reciprocal covering (='decken'), namely 'deckungsgleich' ('gleich'= 'equal'). In congruence geometry, different from transformation geometry, the specific form of these motions is not essential at all. So the students need not, cannot and, in fact, do not memorize them, and motions are not likely to turn into mental obstacles against viewing congruence as an interrelation between two stationary geometrical forms.

For *functional reasoning* in geometry and other mathematical disciplines, like calculus, there is needed a different, and slightly more abstract, concept of motion: What happens in the range of a function, if one 'walks' around in its domain? Example: The area function assigns to each triangle of the Euclidean plane its area. Starting with one triangle, one changes one of its vertices, and one observes, how the area changes. – The metaphorical character of this situation is obvious: There is a space (the domain, i.e. the set of all triangles) and someone or something (the variables) who 'walks' around; and the motions of this someone or something are transferred by some mechanism, like an abstract pantograph, into another space (the range, i.e. the positive real numbers).

One more example, where 'dynamic' imagery and understandings seem to be not helpful for basic concept formation, is the *concept of sequence and limit:* Many students have the wrong idea that a mathematical sequence would possess a last element (with the number ∞) or that one could at least reach such an element (whatever the notion of reaching should mean). – The ground for this misconception is often laid in mathematics teaching itself, e.g. when determining the number π by an approximation:

The students consider a sequence of polygons which have more and more vertices until they finally turn into a circle. Even if the teacher carefully avoids such a wrong diction, the students still can easily get the impression that the circle would be the last element of that sequence: Firstly, because of optical reasons, and secondly, because the aim of the lesson *is* to determine a limit by a sequence of elements, and all the activities evoke this impression, whether the teacher expresses verbally that the limit cannot be reached, or not. Even if the students accept that it is impossible to reach it, they tend to ground the impossibility on limited time and limited arithmetic of human or electronic calculators.

Another example, which is dealt with in the curriculum even earlier, is the decimal fractions of *rational numbers*. The students prove, e.g., that $\frac{1}{3} = 0{,}333\ldots$. The teacher states that the equality holds if there are infinitely many digits '3', and finally, in Germany, there is written $\frac{1}{3} = 0,\overline{3}$ as abbreviation. By this notion the double nature of the concept of limit is expressed. The symbol 'lim...' stands for a request to run through a process and, at the same time, for the result of this process.

For an *algebraic* term, like a+b , this double nature (to be a request for some activity and to be the result of this activity) is well known and useful, but it fails when the activity includes some infinite process. So the students are right, when they refuse to accept the correctness of the equality $0,\overline{3} = \frac{1}{3}$ and all the more $0,\overline{9} = 1$. They take the dynamic part of the double nature of limit seriously (because this part, grounded on the didactical principle of supporting 'dynamical' thinking, is always stressed), and they correctly deny that running through that infinite process, for which an expression like $0,\overline{9}$ stands, will result in the limit. Fischbein (1989) observed that students even deny the symmetry of the equality sign, as they accept $\frac{1}{3} = 0,\overline{3}$, because this expression can be read from left to right, and the digits on the right can be written down one after the other, whereas they refuse $0,\overline{3} = \frac{1}{3}$, because one can never have on the left side all the needed ingredients to produce the result, one never comes to an end and one is not able to say " $= \frac{1}{3}$ ".

The place of the conception of BIU in mathematics didactics

In all times, all over the world, mathematics educators reflected and still do reflect on basic imagery and understandings for mathematical concepts, though they usually do not name them like that and possibly have different or no conceptual frames. There is still missing a theory unifying the relevant disciplines 'mathematics', 'epistemology' and 'psychology'. The work of vom Hofe and my work is one attempt. But the realization in didactical and teaching practice is at least as important as the theory. Which basic imagery and understandings do we think to be adequate? How can we support the students generating adequate basic imagery and understandings? Which inadequate basic imagery and understandings can occur? How are they caused? How can they be improved or corrected? – In my opinion, these are fundamental questions of mathematics education. These questions, as well as my attempt to give preliminary and local answers, are closely connected with Ernest's (1994a) ideas of social constructivism, but in my opinion they are also compatible with more radical variants of constructivism.

References

Bender, P. (1982). Abbildungsgeometrie in der didaktischen Diskussion. *Zentralblatt für Didaktik der Mathematik 14*, 9-24.

Bishop, A. J. (1985). The Social Construction of Meaning – A Significant Development for Mathematics Education. *For the Learning of Mathematics 5* (1), 24-28.

Bosshardt, H.-G. (1981). Vorstellungen – Grenzgänger zwischen Wissenschaft und Alltagsverstand. *Zeitschrift für Semiotik 3*, 363-370.

Campbell, S. and Dawson, A.J. (1995). Learning as Embodied Action. In R. Sutherland and J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education.* Berlin: Springer.

Clark, H. H. and Carlson, T.B. (1981). Content for Comprehension. In J. Long and A.D. Baddeley (Eds.), *Proceedings of the International Symposium on Attention and Performance IX.* Hillsdale: Lawrence Erlbaum.

Davis, E. J. (1978). A Model for Understanding in Mathematics. *Arithmetic Teacher 26* (September), 13-17.

Davis, R. B. and McKnight, C. (1980). The Influence of Semantic Content on Algorithmic Behavior. *Journal of Mathematical Behavior 3* (1), 39-87.

Dörfler, W. (1996). Means for Meaning. To appear in: *Proceedings of ICME 8.*

Engelkamp, J.(Ed.) (1984). *Psychologische Aspekte des Verstehens.* Berlin: Springer.

Ernest, P.(1994a). What is Social Constructivism in the Psychology of Mathematics Education? *Proceedings of the 18th Conference on Psychology of Mathematics Education, Vol. 2.* Lisboa.

Ernest, P. (1994b). Constructivism: Which Form Provides the Most Adequate Theory of Mathematics Learning? *Journal für Mathematik-Didaktik 15*, 327-342.

Fischbein, E. (1987). *Intuition in Science and Mathematics.* Dordrecht: Reidel.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning. *For the Learning of Mathematics 9* (2), 9-14.

Halmos, P. R. (1981). Does Mathematics Have Elements? *The Mathematical Intelligencer 3*, 147-153.

Herscovics, N. and J. C. Bergeron (1983). Models of Understanding. *Zentralblatt für Didaktik der Mathematik 15*, 75-83.

Hörmann, H. (1976). *Meinen und Verstehen.* Frankfurt: Suhrkamp.

Johnson, M. (1987). *The Body in the Mind: The Bodily Basis of Meaning, Imagination and Reason.* Chicago: University of Chicago Press.

Kaput, J. J. (1979). Mathematics and Learning: Roots of Epistemological Status. In: J. Lochhead and J. Clement (Eds.), *Cognitive Process Instruction. Research on Teaching Thinking Skills.* Philadelphia: The Franklin Institute Press.

Krummheuer, G. (1989). Die Veranschaulichung als "formatierte" Argumentation im Mathematikunterricht. *mathematica didactica 12*, 225-243.

Malle, G. (1988). Die Entstehung neuer Denkgegenstände – untersucht am Beispiel der negativen Zahlen. In: W. Dörfler (Ed.), *Kognitive Aspekte mathematischer Begriffsentwicklung.* Wien: Hölder-Pichler-Tempsky.

Mann, I. (1991). *Ich war behindert an Hand der Lehrer und Ärzte.* Weinheim: Beltz.

Papert, S. (1980). *Mindstorms. Children, Computers, and Powerful Ideas.* New York: Basic Books.

Presmeg, N. C. (1994). The Role of Visually Mediated Processes in Classroom Mathematics. *Zentralblatt für Didaktik der Mathematik 26*, 114-117.

Schipper, W. (1982). Stoffauswahl und Stoffanordnung im mathematischen Anfangsunterricht. *Journal für Mathematik-Didaktik 3*, 91-120.

Schreiber, A. (1983). Bemerkungen zur Rolle universeller Ideen im mathematischen Denken. *mathematica didactica 6*, 65-76.

Sheehan, P. W. (Ed.) (1972). *The Function and Nature of Imagery.* New York: Academic Press.

Simon, H. A. (1969). *The Sciences of the Artificial.* Cambridge, MA: MIT Press.

Skemp, R. R. (1976): Relational Understanding and Instrumental Understanding. *Mathematics Teaching 77*, 20-26.

vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte.* Heidelberg: Spektrum.

vom Hofe, R. (1996). On Concepts of the Generation of Adequate Images and Basic Ideas – Normative, Descriptive and Constructive Aspects. To appear in: *What Is Research in*

Mathematics Education, and What Are Its Results? Washington, D.C.: Kluwer.

von Glasersfeld, E. (1989). Constructivism in Education. In: T. Husen and T.N. Postlethwaite (Eds.): *The International Encyclopaedia of Education. Supplementary Volume.* Oxford: Pergamon Press.

Context as Construction

David Clarke and Sue Helme
University of Melbourne, Australia

What is context and what is construct? If knowledge is actively constructed by the cognising learner, and this process of cognition serves the learner's organization of the experiential world, then the learner's experience of situation and context is a consequence of a process of construal. What then of context? Is context an individual construction? The location of mathematical tasks in meaningful contexts for either instructional or assessment purposes seems an obvious development in our continuing attempts to engage the student's interest, and in our recent recognition of the importance of acknowledging and utilising the situated nature of learning.

This paper establishes three key distinctions central to the relationship between context and constructivism:

1. Situation and Context.
It is useful to distinguish the social "situation" which the classroom task offers a student from the personal "context" for that task which is a product of individual construal and ultimately a personal construction of the student.

2. Figurative and Social Contexts.
Every task is encountered within a "social context". Within the social context, we wish to draw attention to the function of the situation described in the task, which we designate the "figurative context". Students appear to attend to this figurative context to different degrees. The degree of significance a student accords the figurative context influences the student's chosen method, the form of their response, and their assessment of the viability of the solution.

3. Abstraction and Contextualisation.
The posing of mathematical tasks in elaborated figurative contexts can be argued on several grounds: understanding, motivation, transfer, and stimulated metacognition. The use of multiple contexts has been advocated as a means of stimulating the

construction by the student of something called "the abstraction". Is our goal to equip learners to disengage mathematics from the contexts in which it might be encountered, or to contextualise the mathematics they are learning in forms which both give it meaning and, ultimately, facilitate its use? It is the suggestion of this paper that it is the contextualisation that is empowering, rather than the possession of some minimally contextualised construct called "the abstraction".

Context in our view is neither a neutral background for the negotiation of mathematical meanings, nor merely a catalyst mediating between task content and the individual's mathematical tool kit. Rather, we should speak of personal task context as an outcome of the realization of the figurative context within the broader social context. This personal task context takes the form of a cognitive representation through which an individual ascribes personal meaning to a task and within which the problem solving process is undertaken.

Introduction

> She was the single artifice of the world
> In which she sang. And when she sang, the sea,
> Whatever self it had, became the self
> That was her song, for she was the maker. Then, we
> As we beheld her striding there alone,
> Knew that there never was a world for her
> Except the one she sang and, singing, made.
> – Wallace Stevens, "The Idea of Order at Key West"

What is context and what is construction? If knowledge is actively constructed by the cognising learner, and this process of cognition serves the learner's organization of the experiential world (Kilpatrick, 1987), then the learner's experience of situation and context is a consequence of a process of construal. That is, all sensory input undergoes a process of construal: a refractive process in which the incoming signal is reconstructed in a form which combines characteristics of the signal and of the medium through which the signal has passed: that is the cognitive framework of the perceiver (Clarke, 1993). The refracted signal is reconstructed as a cognitive representation of the original signal. What then of context? Is context also an individual construction?

Why "meaningful contexts"?

The location of mathematical tasks in meaningful contexts for either instructional or assessment purposes seems an obvious development in our continuing attempts to engage the student's interest, and in our recent recognition of the importance of acknowledging and utilising the situated nature of learning (Lave, 1988). From this perspective, learning is facilitated where students are able to find points of connection between their own experience and what they are trying to understand (Belenky, Clinchy, Goldberger, and Tarule, 1986).

The use of elaborated task contexts in mathematics instruction has been argued on at least three grounds:

- The encountering of new mathematical content in a familiar context facilitates student understanding of the new content since individual representations of that content can be constructed using contextual elements already present in the learner's cognitive framework;
- The utilisation of meaningful contexts facilitates student engagement with the problem, and enhances motivation;
- The learning of mathematical content in familiar contexts holds the promise of increased transfer, application or use of that content in other contexts within the student's immediate or anticipated experience.

Research reported in this paper also suggests that the contextual detail within a task can provide students with metacognitive cues for tool selection and for the evaluation of solution viability based in the student's knowledge of the task situation (the "figurative context") rather than in the student's knowledge of the mathematics (or of the conventions of mathematics problems).

Some recent curriculum initiatives (for example, the *Maths in Context Project*, Romberg, Allison, Clarke, Clarke, Pedro, and Spence, 1991) have given prominence to the idea of "meaningful contexts for learning" as though the context were immutable and interpreted in some universal fashion. By contrast, some theorists within the mathematics education community view the meanings of the mathematics and the associated social exchanges as problematic or open to interpretation. We would like to see *context* discussed from a similar interpretative perspective.

For any contextually-specific instructional material, there will be some students to whom the context does not appeal or is simply foreign (Burton, 1993). Even for those students who find interest in the problem context, there will be some for whom the demands of language or contextual detail constitute an excessive cognitive load. Some curriculum developers have shown a high degree of sensitivity to the personal experiences and values of the students they anticipate interacting with their curriculum materials. A very good example of such an approach can be found in the recent development of calculus materials (Barnes, 1993), in which a structured and considered attempt was made to identify a much wider range of contexts than has been traditionally used in the teaching of calculus. The use of social issues as a context for mathematics instruction has also been specifically advocated (Lovitt and Clarke, 1988), although here the justification is the demonstration of the role of mathematics in enabling students both to understand their world and to assert some degree of control over it.

Even context has to be construed.

The question still remains as to the form in which these task contexts are construed by an individual student. It is useful to distinguish the "situation" which the classroom task offers a student from the personal "context" for that task which is a product of individual construal and ultimately a personal construction of the student. As a consequence of this distinction, the teacher or the curriculum developer cannot offer the student pre-fabricated contexts. Instead, the teacher (or curriculum developer) through the selection of a task, creates a classroom situation defined in the social enaction of the task. These socially defined *situations* are then the basis for the student's construction of *context*. This distinction requires an example.

> In the lesson "How many people can stand in your classroom?" (Lovitt and Clarke, 1988), students are asked to imagine that their classroom is part of a sports stadium. They then confront the problem of how many "standing room" tickets could be sold at a sporting event in an area the size of their classroom. Children work in small groups to devise and implement strategies for solving the problem. Some students, with extensive experience of sporting venues, apply this experience by setting realistic conditions on the amount of space to be allocated to each individual. These students construe the situation as a context of personal significance and familiarity. Students without this experience are less likely to set such "realistic" conditions and tend to locate the task purely within the classroom. While all students participate in the social enaction of the task - that is, all students experience a similar *situation* - individual students construe the *context* of the task in ways which reflect their personal histories.

The situated nature of learning is not being contested in this paper; classroom tasks establish situations for learning. We propose, however, that *context* is an individual construction in response to these socially-enacted *situations*.

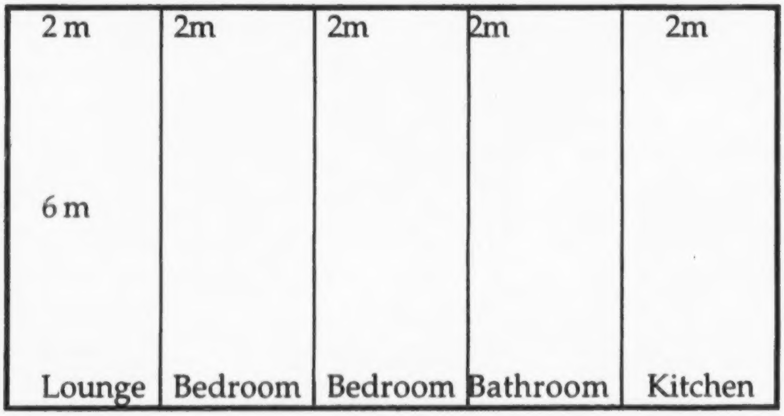
Individuals' constructions of context

In examining individuals' responses to contextualised mathematical tasks, the significance of individual familiarity and personal experience with a context is apparent. Other factors are also evident. Mathematics educators continue to present mathematical tasks in written form, and an emphasis on problem context requires that elaborate detail be provided to establish that context. This detail carries its own dangers. Not only may the context be less than familiar to some students, but the task remains, as always, mediated by the language in which it is framed. This language may require a vocabulary beyond the capability of many students, who in everyday life may interact with similar "meaningful contexts" to those described in the problem situations but do so without the obligation to either encounter the situation in purely written form or to respond to the situation in written form.

The examples which follow are illustrative of the variation in individuals' construal of task purpose and of the acceptable form a solution might take. Differences are evident in the significance accorded the figurative context provided within each problem. In the first example, Alan, a 40-year old music teacher was presented with the following task.

> Freda's flat has five rooms. The total floor area is 60 square metres. Draw a possible plan of her flat. Label each room and show the dimensions (length, width) of each room.

Alan's plan looked like this:



Alan reported that he was not trying to draw a realistic plan. That was not what he thought the task was asking him to do.

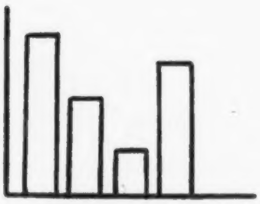
> "I was just thinking of the maths involved. I guess that probably comes from the way you learn maths, just doing calculations and learning calculations. But if I was planning an extension for the house ... I wouldn't think in that kind of way."

Margaret, in the example below, attended similarly to the "school maths" character of the task and the situation in which it was encountered. In fact, the task cues to which Margaret responded were founded exclusively in her experience of school mathematics and made limited use of the intended real-world context portrayed in the task.

> *Task:* Tom is tiling the wall above his kitchen sink. The area to be tiled is 165 *cm* long and 45 *cm* high. The tiles are 15 *cm* by 15 *cm*. How many tiles will he need?
>
> "I'm sure there's a mathematical way of working it out. I'm just trying to work out how you're supposed to do it. You're presumably meant to multiply that by that, and that by that, and divide that into that. I can't remember doing that one at school, but presumably we did it." (Margaret, a 41 year-old social worker)

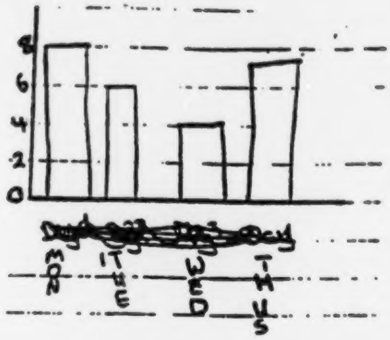
An alternative to offering students "prefabricated task contexts" is to allow students to determine some of the contextual features of the task themselves. In the following example the student was required explicitly to construct a suitable figurative context:

> *Task:*
> a) What might this be a graph of?



> b) Label the graph appropriately.
>
> c) What information is contained in your graph?

The response of one Year 10 student to this task is reproduced below:



> This graph represents how many times David got bashed up by the school bully each day of one week.
>
> On Monday, after a bad weekend, the bully bashed up David 8 times. David always had bad luck on Mondays.
> On Tuesday, after the bully's anger had died, he was merciful and only bashed David up 6 times.
> On Wednesday, the bully was away for half the day, so David was only bashed up 4 times.
> On Thursday, after a bad test result, the bully took out his anger on David, which explains the sharp increase to 7 times.
> There is no column for Friday because David was in hospital.

With the unlabelled graph as stimulus, the student has generated a context for the graph which has personal meaning, and has articulated the meaning of the newly-conceived graph in a manner that is mathematically consistent with the original, and which locates the subject matter of the graph firmly in the world of the student.

The interaction between respondent and task context evident in these excerpts has been addressed in various studies (see, for instance, Clarke, Sullivan and Spandel, 1992). This interaction remains a concern for those responsible for the development of assessment techniques or tools which aim to monitor student learning in relation to either a community of peers or in relation to the goals of a mathematics curriculum. The distinction between the *situation* and the *context* of a mathematics task has already been made. In this section, we have suggested that, for the purposes of assessment, two approaches may minimize the distorting effects of this respondent-task interaction: one is to increase the respondent's control over the identification of the problem context; the other is to offer the respondent multiple contexts and modes of expression through which to demonstrate their understanding. These two alternatives serve to emphasise again the dualism in our use of the term "context": the context in which the task is encountered; and the context depicted in the task. This dualism requires explicit discussion.

Locating the dominant context

> Are you the leaf, the blossom or the bole?
> O body swayed to music, O brightening glance,
> How can we know the dancer from the dance?
> – W. B. Yeats, "Among School Children"

There are two classes of context at work here: the context in which the task is encountered: the "interactive context"; and the "figurative context", the situation described in the task.

The interactive context, for both examples cited above, was the situation of being asked to solve some problems in a relaxed and friendly, but obviously contrived and test-like situation. The interactive context evokes a range of individual perceptions of the requirements of the task. For Alan, only the interactive context was of significance. Yet, some people fully engaged with the task as though its resolution related to some personal situation beyond the interactive context; that is, they engaged significantly with the figurative context. What is the significance of this difference in the way in which different respondents interact with the task context?

If a person in designing Freda's flat asks themselves the question, "Should I really have to go through the bathroom to get from the kitchen to the living room?", are they engaging in more fruitful mathematical activity, more powerful mathematical thinking, higher order problem solving, or are they just embellishing their diagram? Have they demonstrated a deeper understanding of the content through their ability to contextualize the problem in detail, or are they, in fact, demonstrating a need to locate every simple mathematical operation in a concrete form? Wilensky (1991), among others, has argued that the identification of an operation or an object as concrete is an indication of sophisticated rather than primitive thinking with respect to that object.

> Outside of school, in the world, our nascent understanding of a new concept, while not usually formal, is often abstract because we haven't yet constructed the connections that will concretize it. The reason we mistakenly believe that we are moving from the concrete to the abstract is that the more advanced objects of knowledge (e.g., permutations, probabilities) which children gain in the formal operations stage are not concretized by most adults. Since these concepts/operations are not concretized by most of us, they remain abstract and thus it seems as if the most advanced knowledge we have is abstract. (Wilensky, 1991, p. 201)

Are some people distracted or constrained from engaging in "real" (that is, abstract) mathematics by the irrelevant detail of context? Or is this the very demonstration of their mathematical capability? Certainly, it appears that people in situations in which a mathematical calculation could serve an immediate and personal purpose show a high level of discretion in determining whether or not to carry out the calculation, and in choosing what information to use or transform in solving the problem.

At times, this participant-context interaction was very visible in the data.

> *Task:* The 500 *ml* bottle of cooking oil is $3.48 and the 750 *ml* bottle costs $5.09. Which is the better buy?

Margaret's immediate response was: "I always get into difficulty with this at the supermarket." She then proceeded to use algebra to solve the problem. In the process of translating the problem to and from symbolic representation, Margaret lost much of the sense of how the algebraic representation related to the problem details.

Sue, in talking to Margaret, said: "If you were actually in a supermarket, how would you do the task?" Margaret then did the task in a completely different way.

> "If it's complicated, for example, 750 compared with 400, I'd really give up .... This one you can get a bit of a gist of .... I'd make that one three fifty [$3.50] and that one five [$5]. If I doubled that one, that would make $7, then perhaps I'd get half-way between $7 and $3.50. I think I'd then be able to guess.
>
> "When I was saying [the answer], I was surprised that the smaller one was better [value]. It's unusual. But I thought, "Oh well, it's a kind of a trick", and I'm not au fait with oil. Particularly olive oil. Because I never buy olive oil. I wouldn't really be able to tell, just by looking at it, which one was more expensive. And I would assume that because quality of olive oil varies, you could have some that was really quite expensive. So double the size would not necessarily mean better value.
>
> "When I did it the way you do it in the supermarket, I realized I was wrong. I doubled this one [500 *ml*] to make it a litre. And then it became obvious that it was more expensive. A litre would be seven dollars. Two dollars more than the seven-fifty *ml*. But there's not two dollars difference between that [pointing at the 500 *ml* bottle] and that [pointing at the 750 *ml* bottle]. (*"There's only a dollar fifty difference"*) Yeah. That would be kind of how I'd work it out. But if it was four hundred and seven-fifty I wouldn't even try to do it."

For Margaret, the figurative context was one which linked closely with her personal experience. Margaret distinguished between figurative and interactive contexts. Her initial response was to the interactive context (that is, to the problem as an example of school mathematics), and drew upon her knowledge of algebra. However, when prompted to relate the task to her personal experience, Margaret responded with two additional distinct methods of solution, and a completely different answer. In this second approach, Margaret employed whole number arithmetic.

In which of the above two responses is Margaret demonstrating the greater mathematical capability? The *National Statement on Mathematics for Australian Schools* suggested that "Students should develop confidence and competence in dealing with commonly occurring situations" (Australian Education Council, 1991, p. 11), and made the additional point:

> Procedures which are very practical in some contexts may be less so in other contexts. For example, while most people may rarely need to find the volume of an irregular solid, a builder may do so frequently in order to estimate the cement needed for foundations. The builder, however, is likely to use a procedure which is quite specific to the constraints of measuring a building site. (Australian Education Council, 1991, p. 12).

As mathematics educators, we need to be aware of the tension between the figurative and interactive contexts.

The construction of context

What was it that triggered Margaret's response? The dominant context for Margaret was determined by her response to the metacognitive question, "Within which context am I most likely to resolve the task?" - In fact, Margaret said, "I always get into difficulties with this at the supermarket", and responded to the task as one located in school mathematics, using algebra because, within this context, algebra was the approach in which she had greatest confidence. Is this deliberate selection of task context common among other individuals? Do others select the task context on the same grounds? What are the factors that affect an individual's contextualisation of a task? Many conventional mathematical tasks, especially those in contemporary textbooks, have been deliberately decontextualized: non-symbolic language and references to any social or practical situations have been minimalised. We would argue that all tasks are socially-situated, but that the task context is a consequence of a process of construal and construction on the part of the individual responding to the task. This process can be seen even when the respondent is confronted with a so-called "context-free" task.

Task: Which is bigger: 5/3 or 7/4?

> This task really distinguished between participants with a strong mathematics background and those without. David, for instance, with little maths background (Lecturer in Art; Year 11, Terminal maths), said,
>
> "My first guess is seven over four is bigger than five over three.
>
> [Tried doubling the denominators and compared five-sixths and seven-eighths, but reached no conclusion. Then tried inverting the fractions and recognized that three-fifths was bigger than four-sevenths, but did not know how to use that result].
>
> I'm just playing with things I'm more familiar with ... It's clear that three-fifths is bigger than four-sevenths. I'm very familiar with smaller digits over larger digits, but not larger digits over smaller ones. Thirds don't have five of them, thirds have three of them!
>
> [Then drew "circle" diagrams to model the three-thirds and the four-quarters].
>
> So what's left over is two-thirds compared with three-fourths, having used up our four-fourths and three of our thirds in our circle. So which is bigger out of two-thirds and three-quarters, so I'd say three-quarters. Seven-fourths is bigger.

Having constructed a context which he found meaningful, David was finally able to solve the problem. In David's case, the context was purely diagrammatic, rather than arising from a social situation or some "practical" application of the mathematical content. Yet, this notional context was both necessary and sufficient to sustain his problem solving. It appears that even "decontextualised" tasks such as "Which is larger: 5/3 or 7/4?" are situated; the tendency in response is to contextualise the task. This is done through the construction of a context which gives meaning to the task and to the problem solving process.

Interpreting context

The title of this section is deliberately ambiguous and doubly appropriate because we must answer the question, "What do we mean by 'context'?" and the question, "By what process is the context in a problem situation interpreted?"

In the small adult sample employed in this study, it was noticable that the least contextualised responses to the "Freda's flat" task came from among the most academically successful adults. For these people, one of the consequences of schooling appeared to be the successful mastery of the rules of the institution. These rules divorce mathematics from any real-world constraints, and endorse minimalist responses, provided these are consistent with the mathematical conditions of the task. It was by appeal to these rules that Alan could design a flat no sane person would consider building, yet feel that he had met the requirements of the task.

> Mathematically it doesn't matter if I do it creatively or not, does it? It's a very boring flat! (Alan)

In framing this response, Alan clearly discarded certain aspects of the figurative context. Alan's choice of problem solving approach required the selective identification of elements within the problem which Alan felt were essential to its solution. From this, we suspect that some of Alan's problem solving time was spent in a process of partial disengagement from context. Central to our argument at this point is the question of whether Alan could be said to have decontextualized the task, that is to have "abstracted" the relevant mathematics.

Schoenfeld (1989), among others, in comparing the problem solving efforts of "expert" and "novice" mathematicians, drew attention to the significantly greater amount of time spent by the experts in analyzing the problem.

> The mathematician spent more than half of his allotted time trying to *make sense of the problem.* (Schoenfeld, 1989, p. 95, *our emphasis*)

One inference that could be drawn from Schoenfeld's discussion (1989, p. 97) is that the process of "making sense" engaged in by the expert problem solver required the abstraction of relevant mathematical concepts and procedures from the problem context, and that it was this process of abstraction which both occupied the "expert" and distinguished expert from novice. Sweller (1992), however, would explain precisely the same phenomenon as occurring through the expert's attempts to identify a correspondence between that problem and previously encountered problems. In this process, similarity to previously encountered problem contexts would be a key determinant of problem solving success. From this second perspective, the expert already possesses a rich set of problem schemas which may match the stimulus provided by the task to a greater or lesser extent. These two models of the problem solving process are very different, and locate mathematical activity as either external to the figurative context or embedded within it.

Both Alan and Margaret appeared to have a strong sense of what it was to do school mathematics successfully. One question here is whether success at academic (school) mathematics represents a valid measure of mathematical power. It appears that school mathematics has successfully represented abstract reasoning as the most sophisticated form of mathematical reasoning. Wilensky (1991) and Clarke, Frid and Barnett (1993) have suggested that this view is mistaken.

> The process which commonly has been associated with this [mathematical disposition] in the past has been that of abstraction of some mathematics from its embeddedness in some problem context. We would suggest rather, that the relevant process should be seen as the learner's previous contextualization of mathematics in an increasing diversity of situations, and that it is this access to multiply-contextualized representations of the relevant mathematics in problematic situations which determines the level of sophistication of the mathematical learning. (Clarke, Frid and Barnett, 1993, p. 9)

In answer to the two questions posed at the start of this section: We have attempted to demonstrate the duality in the way in which context is, and must be, interpreted; we have further suggested that context is interpreted via a two-step process involving the construal of the socially-situated task by each individual and the consequent construction of a personal task context by which the individual gives meaning to the task and, in so doing, facilitates the problem solving process.

Conclusion

In this paper we have attempted to establish three dualist perspectives on context.

Perspective 1. Situation and Context.
It is useful to distinguish the "situation" which the classroom task offers a student from the personal "context" for that task which is a product of individual construal and ultimately a personal construction of the student. We would argue that all tasks are socially-situated, but that the task context is a consequence of a process of construal and construction on the part of the individual responding to the task, resulting in the construction of a *personal task context* by which the individual gives meaning to the task and, in so doing, facilitates the problem solving process.

Perspective 2. Interactive and Figurative Contexts.
In the process of context construal and construction, there are two classes of context with which the individual must interact simultaneously: the context in which the task is encountered, which we designate as the "interactive context"; and the "figurative context", that is, the situation described in the task. An individual's response to a mathematical task will inevitably reflect the dynamic between these two classes of context. Research could usefully address the following two questions:

- What is it that establishes either the interactive or figurative context as dominant, and what forms are taken by the resultant personal task context as a consequence of that dominance?

- What role is played by the dominance of interactive or figurative contexts in facilitating construction of personal task contexts associated with effective learning?

Perspective 3. Abstraction and Contextualisation.
Our final distinction draws attention to those characteristic behaviours or mental processes which we associate with successful problem solvers in mathematics. Is it our goal as mathematics educators to equip learners to solve problems by disengaging mathematics from the contexts in which it is encountered, or to contextualise the mathematics they are learning in forms which both give it meaning and, ultimately, facilitate its use? This issue of the goal of mathematics education is central to our use of context in instruction.

Constructivism renders our relationship with context both a dynamic and a problematic one. Constructivism offers a model of cognition as a self-regulatory, dynamic process, whereby reality (or, more precisely, the learner's knowledge as reality) is constructed intersubjectively through social negotiation. Context in this view is neither a neutral background for the negotiation of mathematical meanings, nor merely a catalyst mediating between task content and the individual's mathematical tool kit. Rather, we should speak of personal task context as an outcome of the interplay between the interactive and figurative contexts, taking the form of a cognitive representation through which an individual ascribes personal meaning to a task and within which the problem solving process is undertaken.

Acknowledgements

The authors would like to thank Isabel Romero for drawing to their attention the poem by Wallace Stevens.

References

Australian Education Council (AEC) (1991). *A national statement on mathematics for Australian schools.* Carlton, Victoria: Curriculum Corporation.

Barnes, M. (1993). *Investigating change.* Carlton, Victoria: Curriculum Corporation.

Belenky, M., Clinchy, B., Goldberger, N., and Tarule, J. (1986). *Women's ways of knowing.* New York: Basic Books.

Burton, L. (1993). Implications of constructivism for achievement in mathematics. In J. A. Malone and P. C. S. Taylor, *Constructivist interpretations of teaching and learning mathematics*. Proceedings of Topic Group 10 at the Seventh International Congress on Mathematical Education, Quebec, Canada, August, 1992. Perth, Western Australia: Curtin University of Technology.

Clarke, D.J. (1993). Unguarded metaphors in educational theorizing: Metaphorically modelling the social-personal interface. Paper presented to the 15th Annual Meeting of the North American chapter of the International Group for Psychology in Mathematics Education (PME-NA), Asilomar Conference Centre, California, USA, October 17 - 20, 1993.

Clarke, D.J., Frid, S., and Barnett, C. (1993). Triadic systems in education: Categorical, cultural or coincidence. Paper presented to the 16th Annual Conference of the Mathematics Education Research Group of Australasia, Brisbane, July, 1993.

Clarke, D.J., Sullivan, P.A., and Spandel, U. (1992). *Student response characteristics to open-ended tasks in mathematical and other academic contexts.* Research Report No. 7. Oakleigh, Vic.: MTLC.

Kilpatrick, J. (1987). What constructivism might be in mathematics education. *Proceedings of the Eleventh International Conference on the Psychology of Mathematics* Education, Montreal, Vol. 1, pp. 3 - 27.

Lave, J. (1988). *Cognition in practice: Mind, Mathematics and culture in everyday life.* Cambridge: Cambridge University Press.

Lovitt, C. and Clarke, D. M. (1988). *The Mathematics Curriculum and Teaching Program. Activity Bank: Volume 2.* Canberra: Curriculum Corporation.

Romberg, T. A., Allison, J., Clarke, B., Clarke, D.M., Pedro, J., and Spence, M. (1991). *A blueprint for maths in context: A connected curriculum for grades 5 - 8.* Madison, WI: Wisconsin Center for Educational Research.

Schoenfeld, A. (1989). Teaching mathematical thinking and problem solving. In L.B. Resnick and L.E. Klopfer (Eds.), *Toward the thinking curriculum: Current cognitive research.* 1989 Yearbook of the Association for Supervision and Curiculum Development.

Sweller, J. (1992). Cognitive theories and their implications for mathematics instruction. In G. Leder (Ed.), *Assessment and* *learning of mathematics*. Hawthorn, Victoria: Australian Council for Educational Research.

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel and S. Papert (Eds.), *Constructionism*. Norwood, NJ: Ablex.

Yeats, W.B. (1961). Among School Children. In *The Collected Poems of W.B. Yeats*. London: Macmillan.

Publikationsserier vid Pedagogiska fakulteten

Beställningsadress: Pedagogiska fakulteten vid Åbo Akademi
Redaktionsrådet
PB 311, 65101 Vasa, Finland
Tel: +358-(0)6-3247 111
Fax: +358-(0)6-3247 302

Rapporter (blå serie)

1/1992 Lindfors Linnéa. Formgivning i slöjd. Ämnesteoretisk och slöjdpedagogisk orienteringsgrund med exempel från textilslöjdsundervisning. Pris 40 mk.

2/1992 Björkqvist Ole (red.). Forskning och utvecklingsarbete i de matematiska ämnenas didaktik. Pris 35 mk.

3/1992 Østern Anna-Lena. Språkglädje och språklig medvetenhet. Träningsprogram för språklig medvetenhet. Pris 20 mk.

4/1993 Grönholm Maija. TV on pang pang - verbisanaston kehitys toisen kielen kirjoittamisessa. Pris 60 mk.

5/1993 Malmberg Lars-Erik. Självuppfattning. Några teoretiska aspekter och empiriska resultat. Pris 20 mk.

6/1993 Myrskog Gunilla. Lärarstuderandes tankar om inlärning. En studie av lärarstuderandes uppfattningar av inlärning i relation till olika kontexter inom lärarutbildningen. Pris 35 mk.

7/1994 Heilä-Ylikallio Ria. Mönster i förskolbarns möte med skriftspråket. Pris 40 mk.

8/1994 Björkqvist Ole, Finne Lena (red.). Matematikdidaktik i Norden. Pris 35 mk.

9/1994 Østern Anna-Lena. "Sol, sol! - Jag vill va´ solen!" Kontextbyggande och perspektivbyten i barns berättande och drama. Pris 40 mk.

10/1995 Sundqvist Rolf. Didaktiskt tänkande. En studie om uppfattningar av undervisning. Pris 35 mk.

11/1995 Stolpe Lisen. Förskola i skola. Förskoleundervisning i ett förändringsperspektiv. Pris 30 mk.

12/1996 Eklund Marita. Samundervisning i gymnastik. En undersökning av gymnastiklärares erfarenheter av och attityder till gemensam undervisning av flickor och pojkar. Pris 40 mk.

13/1996 Ström Kristina. Lärare, försvarsadvokat, lindansare eller .... Speciallärarens syn på sin verksamhet och roll på högstadiet. Pris 40 mk.

14/1997 Lindfors Linnéa. Slöjdfostran i kulturkampen. Del IV. Positionsbestämning av slöjdpedagogisk forskning. Analyser av lärdomsprov vid Åbo Akademi. Pris 40 mk.

15/1997 Nygren-Landgärds Christina. Slöjdfostran i kulturkampen. Del VI. Det tänkta och det upplevda undervisningsämnet. En studie om undervisning i slöjdpedagogik. Pris 35 mk.

16/1997 Hansén Sven-Erik. "Jag är proffs på det här". Om lärarens arbete under en tid av förändring. Pris 37 mk.

Publikationer (grön serie)

1/1992 Stormbom Jarl. Läraren, professionen och didaktiken - några observationer i ett både teoretiskt och praktiskt problem. Pris 35 mk. Slutsåld.

2/1993 Uljens Michael. Den pedagogiska flugan - en analysmodell för didaktiskt handlande. Pris 10 mk.

3/1993 Ström Kristina. Speciallärarnas arbete på högstadiet. Pris 25 mk.

4/1993 Lindfors Linnéa. Slöjdfostran i kulturkampen. Del I. Studier i den finländska skolslöjdens läroplaner 1912-1994. Pris 35 mk.

5/1993 Myrskog Gunilla. Lärares tankar om studerandes inlärning. En studie rörande uppfattningar av studerandes inlärning bland lärare vid Vasa svenska hälsovårdsläroanstalt. Pris 25 mk.

6/1993 Wörlund Kate. Modeller för integrerad specialundervisning i finlandssvenska högstadier. Pris 35 mk.

7/1993 Vestergren Lilian. Logiskt tänkande och begåvning - effekten av träning av logiskt tänkande och problemlösning samt en analys av kvalitativa skillnader i tänkande och tankestrategier hos barn. Pris 25 mk.

8/1994 Björkqvist Ole. Utvärdering av matematikkunskaperna i årskurs 9 i grundskolan. Pris 20 mk.

9/1994 Uljens Michael (red.). Studier i inlärning, undervisning och utvärdering. Pris 30 mk.

10/1994 Finne Lena. Det matematiska klassrummet. Pris 15 mk.

11/1994 Sundholm Mikael & Finne Lena. Det globala klassrummet. Pris 38 mk. Slutsåld.

12/1994 Østern Anna-Lena, Backas Ingegerd, Sarling Isabella. Dramatisk improvisation. Ledarens metodik och pedagogiska grundsyn. Pris 30 mk.

13/1994 Østern Anna-Lena (red.). Evaluering i modersmålet - nordiskt perspektiv. Pris 40 mk. Slutsåld.

14/1995 Eriksson Anna-Lena, West Anna-Lena, Hannus-Gullmets Britta. Dövskolan i Borgå 1946-1993. Pris 25 mk.

15/1995 Palmberg Irmeli. Natur och miljö i didaktiskt perspektiv. En bibliografi. Pris 28 mk.

16/1995 Björkqvist Ole. Utvärdering av matematikkunskaperna i årskurs 9 i grundskolan, del II. Pris 20 mk.

17/1995 Uljens Michael. Att förklara utbildningsplaner. Två strukturekvationsmodeller analyserade med LISREL. Pris 20 mk.

18/1995 Grönholm Maija. Landsomfattande prov i finska i årskurs 9. En utvärdering av de svenska högstadiernas resultat våren 1994. Pris 20 mk.

19/1995 Malmberg-Tulonen Eva. Slöjdfostran i kulturkampen. Del III. Skolslöjdens betydelse ur elevperspektiv. En lokal undersökning. Pris 25 mk.

20/1995 Lindfors Linnéa. Slöjdfostran i kulturkampen. Del II. En översikt av slöjdpedagogikens utvecklingshistoria i ett finländskt perspektiv. Pris 30 mk.

21/1996 Grönholm Maija. Landsomfattande prov i finska i årskurs 6. Pris 25 mk.

22/1997 Østern Anna-Lena (red.). Att växa som handledare. Studier i utveckling av professionell kompetens. Pris 40 mk.

23/1997 Björkqvist Ole. Utvärdering av matematikkunskaperna i årskurs 7 i grundskolan. 25 mk.

24/1997 Hansén Sven-Erik (red.). Att söka och undersöka. Om utbildning, undervisning och ämnesinnehåll. Pris 30 mk.

25/1997 Østern Anna-Lena (red.). Unga röster. Om undervisning i modersmål och litteratur i finlandssvenska högstadier och gymnasier. Pris: 40 mk.

Studie- och undervisningsmaterial (grå serie)

1/1992 Østern Anna-Lena. Språkglädje och språklig medvetenhet. Uppleva, tala, tänka. Lekar och övningar för en- och flerspråkiga barn i åldern fem till åtta år. Pris 35 mk.

2/1993 Stolpe Lisen. Förskoleundervisning i grundskolan. Mål, innehåll, verksamhet. Handledningsdokument för planerare. Pris 25 mk. Slutsåld.

3/1993 Palmberg Irmeli. Cytologi-genetik-evolutik. Övningar. Pris 30 mk.

4/1993 Malmberg Lars-Erik. Fostran och utbildning i samhälle. En utbildningssociologisk betraktelse. Pris 30 mk.

5/1993 Østern Anna-Lena. Pedagogiskt drama - en didaktisk handledning. Pris 30 mk. Slutsåld.

6/1993 Wigell-Kevin Ringa-May. Idrottens sociala fält. Pris 30 mk.

7/1994 Burman Lars. Planering och utvärdering i matematikundervisningen. Pris 30 mk.

8/1994 Palmberg Irmeli. Botanikens grunder. Pris 40 mk.

9/1994 Palmberg Irmeli. Uteundervisning i vinter- och vårbiologi. Pris 30 mk.

10/1995 Nyberg Rainer. Att skriva avhandling. Guide för pro gradu skribenter i pedagogik. Pris 25 mk. Slutsåld.

11/1995 Österberg Frey, Österberg Leif, Vidberg Henrik, Rudnäs Per-Erik, Mitts Harry, Finne Lena, Dahlbacka Jarl, Burman Lars, Björkqvist Eivor, Björkqvist Ole, Axell May-Gret. Verklig matematik. Pris 33 mk.

12/1996 Lithén Hans-Olof. Ornitologi för grundskolan. Pris 35 mk.

13/1996 Burman Lars. Individuell problemlösning i klassen. Matematikproblem för systematisk övning av problemlösning i högstadiet. Pris 38 mk.

14/1997 Ria Heilä-Ylikallio & Anna-Lena Østern. Storyline och drama som analysredskap vid arbete med Yvonne Hoffmans bok De fyra portarna. En didaktisk handledning. Pris 30 mk.

15/1997 Kristina Stenros & Berit Kurtén-Finnäs. Laborationer i miljökunskap. Pris 30 mk.

16/1997 E.A. Gamini Fonseka. Companion to Drama. Pris 35 mk.

17/1997 Burman Lars. Mera problemlösning i klassen. Matematikproblem för systematisk övning av problemlösning i högstadiet. Pris 30 mk.

18/1997 Lars Burman & Stefan Röj. Matematikprojekt i högstadiet. Tips och underlag för genomförande av matematikprojekt i högstadiet. Pris 35 mk.

19/1997 Anna-Lena Østern & Ria Heilä-Ylikallio. Storyline och drama som analysredskap vid arbete med Yvonne Hoffmans bok Kniven och ringen. En didaktisk handledning. Pris 25 mk.

20/1997 Anna-Lena Østern. Att läsa medietext. En introduktion till mediepedagogiken. Pris 25 mk.

Series from the Faculty of Education

Available from:
Faculty of Education
Åbo Akademi University
PO Box 311
65101 Vasa, Finland
Phone: +358-(0)6-3247111
Fax: +358-(0)6-3247302

Reports (blue series)

1/1994 Lars-Erik Malmberg. Future-orientation of Finnish Students. Price FIM 30.

2/1997 Mia Porko. Craft Education in the Cultural Struggle Part V. Students´ conceptions of sloyd in the comprehensive school. An attempt to apply the phenomenographic method in research on sloyd education. Price FIM 25.

Publications (green series)

1/1996 Lars-Erik Malmberg (Ed.). Initiation of a Teacher Education Project in Tanzania (TEPT). Price FIM 25.

2/1997 Làzaro Moreno Herrera. Craft Education in the Cultural Struggle. Part VII. Cuban Sloyd. History and educational research on developing creativity. Four articles. Price FIM 27.

ISBN 952-12-0151-7
ISSN 1236-0856

Available from:
Faculty of Education
Åbo Akademi University
PB 311,
65101 Vasa, Finland.
Phone: +358-(0)6-3247111
Fax: +358-(0)6-3247302

Blue series: Reports from the Faculty of Education
Green series: Publications from the Faculty of Education

BEST COPY AVAILABLE